The strong containment lattice of Schunck classes of finite soluble groups

Andrew Philip Wilson

A thesis submitted for the degree of
Doctor of Philosophy at the
University of Warwick

Mathematics Institute
University of Warwick

December 1985

Contents

Acknowledgements

I gratefully acknowledge the financial support of the Science and Engineering Council.

Thanks also to my supervisor, Dr. Trevor Hawkes and to all my friends at Warwick.

Finally, to Alison, thank you for all the love, support and understanding you've given over recent months.

Declaration

All the work in this thesis is my own unless stated otherwise. No part of it has appeared elsewhere.

Summary

This thesis is an investigation into some of the lattice properties of the strong containment lattice $(\underline{H}, <<)$ of Schunck classes and also of its important sublattice $(\underline{D}, <<)$.

The general aim is to characterise lattice properties of Schunck classes by avoidance class properties. Our main result, Theorem 8.5, is an avoidance class characterisation of those $\underline{D}$-classes all of whose maximal ascending proper chains of $\underline{D}$-classes to $\underline{\underline{S}}$ have the same length. The problem extended to $\underline{H}$ is much more difficult but in Corollary 4.3 we describe an avoidance class condition for a Schunck class only to have chains of finite length to $\underline{\underline{S}}$.

The lack of duality in $\underline{H}$ shows up clearly in section 3. The fascinating problem of deciding whether or not $\underline{H}$ is atomic is considered in section 9. Our results suggest that it probably is since any counterexample must be very complicated.

Notation

All groups considered will be assumed to be finite and soluble.

Let G and H be groups and let U and V be kG-modules for some field k. Let G act on a set X. Let π be a set of primes and p a prime. Let m and n be natural numbers. Let $\underline{\underline{X}}$ be a class of groups.

$H \leq G$	H is a subgroup of G.
$H < G$ (or $H \lneq G$)	$H \leq G$ and $H \neq G$.
$H \trianglelefteq G$	H is a normal subgroup of G.
H char G	H is a characteristic subgroup of G.
$H <\cdot\, G$	H is a maximal subgroup of G.
$H \cdot\!< G$	H is a minimal subgroup of G.
$Z(G)$	The centre of G.
G'	The derived subgroup of G.
$Soc(G)$	The direct product of all minimal normal subgroups of G.
$Core_G(H)$	The largest normal subgroup of G contained in H where $H \leq G$.
H^G	The smallest normal subgroup of G containing H. (The normal closure of H in G).
$O_\pi(G)$	The maximal normal π-subgroup of G.
$Syl_p(G)$	The set of Sylow p-subgroups of G.
$Stab_G(X)$	$\{g \in G : X^g = X\}$ where $(x,g) \to x^g$ is the action of G on X.

$C_G(H)$	The centralizer of H in G where $H \leq G$.
$C_G(X)$	The centralizer of X in G , $= \{g \in G : x^g = x$ for all $x \in X\}$.
Ker(G on V)	$C_G(V)$.
V_H	V viewed as an H-module by restriction where $H \leq G$.
$\mathrm{Hom}_{kG}(U,V)$	The kG-homomorphisms from U to V .
Aut(H)	The automorphism group of H .
$G \sim H$	The regular wreath product of G by H .
GF(p)	The field with p elements.
S_n	The symmetric group on n letters.
A_n	The alternating group on n letters.
Z_n	The cyclic group of order n .
$E(n/p)$	The semi-direct product of Z_n with a faithful, irreducible Z_n-module over GF(p).
$\mathbb{P}$	The set of prime numbers.
π'	$\mathbb{P} \setminus \pi$
$\underline{\underline{S}}$	The class of soluble groups (finite).
$\underline{\underline{P}}$	The class of primitive groups.
$\underline{\underline{I}}$	The class of groups of order one.
$\underline{\underline{S}}_\pi$	The class of groups of order a π-number.
$\underline{\underline{N}}$	The class of nilpotent groups.
$\underline{\underline{A}}$	The class of abelian groups.

$\lvert \underline{\underline{X}} \rvert$	The number of isomorphism classes of $\underline{\underline{X}}$-groups.
$m \mid n$	m divides n .
$m \nmid n$	m does not divide n .
(m,n)	The greatest common divisor of m and n .
$[G,H]$	The commutator subgroup of G and H for subgroups G and H of a group L .
$\Gamma_n(G)$	$[\underbrace{G,G, \ldots,G}_{n}]$. $(\Gamma_1(G) = G)$.
$\lvert G:H \rvert$	The index of H in G .
$Q(G)$	The set of groups isomorphic to a quotient of G .
$(Q-1)(G)$	$\{H \in Q(G) : \lvert H \rvert < \lvert G \rvert\}$.
$\langle G_\lambda : \lambda \in \Lambda \rangle$	The subgroup of G generated by a set $\{G_\lambda : \lambda \in \Lambda\}$ of subgroups of G .

Introduction

The study of Schunck classes emerged from work by Gaschütz as he tried to describe both Sylow subgroups and Carter subgroups as part of a more general theory. Schunck characterized those classes of groups $\underline{\underline{X}}$ for which every group (always assumed to be finite and soluble) has at least one $\underline{\underline{X}}$-group with certain Sylow type properties.

Doerk and Hawkes showed that, with respect to the partial ordering of strong containment, the family of all Schunck classes form a lattice $\underline{H}$ when suitable definitions of meet and join are adopted. It is this lattice which we discuss here.

An important sublattice, denoted by $\underline{D}$, of $\underline{H}$ was discovered by Wood. $\underline{D}$-classes are Schunck classes which determine in each group a set of subgroups with properties closer to those of Sylow subgroups than general Schunck classes do.

We begin in section 1 of this work with a brief summary of some of the standard group theoretical results we need in our later discussions. Section 2 deals with basic properties of Schunck classes and the lattice $\underline{H}$.

In Section 3 we explore two constructions which yield Schunck classes from any class of primitive groups. One is very simple and the other, introduced by Hawkes in [6], can be modified slightly, as in 5.4, to yield $\underline{D}$-classes. With only a few easy results we are able to characterize the meet irreducible Schunck classes in $\underline{H}$.

In Section 4 the aim is to characterize those Schunck classes all of whose ascending chains have finite lengths.

The next four sections contain results concerning $\underline{D}$-boundaries and methods of construction. We also prepare for the main result of this thesis namely Theorem 8.5.

In Section 9 we tackle the problem of deciding whether or not $\underline{H}$ is atomic.

§1. Group Theory.

Throughout this thesis all groups will be assumed to be finite and soluble even though some of the results remain true in a larger universe.

A. The Fitting subgroup and nilpotent length.

We call the unique maximal normal nilpotent subgroup of a group G the *Fitting subgroup* and denote it by $F(G)$. For the existence of such a subgroup see Huppert [8] III.4.2. For a group G we define subgroups $\{F_i(G): i = 1,2,....\}$ by $F_1(G) = F(G)$ and $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$. This yields a characteristic series for G, $1 < F(G) < F_2(G) < ..... < F_k(G) = F_{k+1}(G) = G$ in which all the factors $F_i(G)/F_{i-1}(G)$ are nilpotent. The smallest number k for which $F_k(G) = G$ is called the *nilpotent length* or *Fitting height*. It is clear that such a number k always exists for a finite soluble group G.

An alternative definition of nilpotent length, equivalent to the former one of course, is described as follows. Let $L(G)$ be the unique minimal normal subgroup of G such that $G/L(G)$ is nilpotent. Then $L(G)$ is called the *nilpotent residual* of G. We set $L_1(G) = L(G)$ and for $i = 2,3,.....$ we define $L_i(G)$ to be $L(L_{i-1}(G))$. This again yields a characteristic series, $1 = L_{k+1}(G) = L_k(G) < L_{k-1}(G) < ..... < L_1(G) < G$, for G. Again the nilpotent length k of G is determined as the smallest number for which $L_k(G) = 1$. We write $\ell(G)$ for the nilpotent length of G .

(1.1) Lemma

Let $\ell(G) = \ell$ for a finite, soluble group G . Then, if $N \trianglelefteq G$, the nilpotent length of G/N is at most ℓ and $\ell(H) \leq \ell$ for each subgroup H of G .

The following result will be very useful to us.

(1.2) Lemma

For a finite, soluble group G , we have $C_G(F(G)) \leq F(G)$.

Proof See Huppert [8] III.4.2. □

(1.3) Lemma

For a finite, soluble group G , the Fitting subgroup is the intersection of the centralizers in G of all the chief factors of G .

Proof Huppert [8] III.4.3. □

B. The Frattini subgroup.

We define the *Frattini subgroup* $\Phi(G)$ of a group G to be the intersection of all maximal subgroups of G. Then $\Phi(G)$ is a characteristic nilpotent subgroup of G (see Huppert [8] III.3.6.) and so $\Phi(G) \leq F(G)$.

If M/N is a chief factor of G, then we call M/N a *Frattini* chief factor if $M/N \leq \Phi(G/N)$, and *complemented* otherwise.

(1.4) Lemma

If $N \trianglelefteq G$ and $N \leq \Phi(G)$, then $\Phi(G/N) = \Phi(G)/N$.

Proof Huppert [8] III.3.4. □

C. Group Actions.

The results in this section are very basic but nevertheless useful to us.

(1.5) Definition

Let G be a group acting on a group H. Then G *stabilizes* a series $1 = H_0 \leq H_1 \leq \ldots\ldots \leq H_n = H$ if $[G,H_i] \leq H_{i-1}$ for $i = 1,\ldots.,n$, where $[g,h] = h^{-g}h$ for $g \in G$ and $h \in H$.

(1.6) Lemma

If H is a π-group and G acts faithfully on H, then G stabilizes a series for H only if G is a π-group.

Proof Gagen [3] Theorem 0.1. □

D. Extraspecial groups.

A p-group G is called *special* if either G is elementary abelian or $\Phi(G) = G' = Z(G)$ is elementary abelian. Hence a special group has class at most two.

An *extraspecial* group G is a non-abelian special group with $|Z(G)| = p$.

(1.7) Definition

Let G_1 and G_2 be groups such that G_i has a normal subgroup N_i contained in $Z(G_i)$ for $i = 1,2$. Suppose α is an isomorphism from N_1 to N_2 . We set $N = \{(n,n^\alpha) : n \in N_1\}$. Then $N \trianglelefteq G_1 \times G_2$. We define $G_1 \curlyvee G_2 = (G_1 \times G_2)/N$ and call $G_1 \curlyvee G_2$ a *central product* of G_1 and G_2 .

(1.8) Lemma

Let G be a non-abelian p-group with cyclic centre and $G/Z(G)$ elementary abelian. Then G is a central product of non-abelian groups with $|G_i/Z(G_i)| = p^2$ and $Z(G_i) = Z(G)$. Furthermore $|G'| = p$.

Proof Huppert [8] III.13.7. □

Notice that, in particular, this decomposition holds for extraspecial groups.

(1.9) Lemma

If G is a non-abelian group of order p^3 and with $|Z(G)| = p$, then G is extraspecial.

Proof Since $|G/Z(G)| = p^2$, it follows that $G/Z(G)$ is abelian and by Huppert [8] III.2.13 the exponent of $G/Z(G)$ is p. Therefore $|G'| = p$ by Lemma 1.8. Since $G/Z(G)$ is elementary abelian, we have $\Phi(G) = Z(G)$ and hence G is extraspecial. □

E. Representation Theory.

This section will be kept very brief and we only include the results which we need later.

(1.10) Lemma

If V is an irreducible $GF(p).G$-module, then $O_p(G) \leq Ker(G \text{ on } V)$.

Proof Huppert [8] V.5.17. □

(1.11) Lemma

Let G be an abelian group. If V is an irreducible $GF(p).G$-module, then $G/Ker(G \text{ on } V)$ is cyclic.

Proof Huppert [8] II.3.10. □

(1.12) Corollary

If G has a faithful, irreducible representation over $GF(p)$, then $Z(G)$ is cyclic. □

The following result contains all that we shall require from Clifford's Theorem.

(1.13) Lemma

Let V be an irreducible $GF(p).G$-module and N a normal subgroup of G . If U is an irreducible N-submodule of V_N , then Ug

is an irreducible N-submodule of V_N for each $g \in G$.

Furthermore, $V_N = V_1 \oplus \ldots\ldots \oplus V_r$ where each V_i is a direct sum of all G-isomorphic copies of an irreducible N-submodule U_i . The V_i's are called *homogeneous components* of V_N . The V_i's are permuted transitively by right multiplication by elements of G .

Proof Huppert [8] V.17.3. □

The next result is extremely important in the construction of primitive groups with interesting properties.

(1.14) Lemma

Let G be a group such that $Soc(G) = N_1 \times \ldots\ldots \times N_t$ where the N_i's are pairwise non-G-isomorphic minimal normal subgroups of G . Suppose that H is a subgroup of G satisfying $H \cap Soc(G) = (H \cap N_1) \times \ldots\ldots \times (H \cap N_t)$. Let p be a prime such that $O_p(G) = 1$ and let U be an irreducible GF(p).H-module such that for all $i \in \{1,\ldots,t\}$ satisfying $H \cap N_i = N_i$, $U_{H \cap N_i}$ is non-trivial.

Then there exists an irreducible GF(p).G-module V , faithful for G and satisfying $U \in Q(V_H)$.

Proof Förster [2] 1.9. □

(1.15) Corollary

Let G be a group having a unique minimal normal subgroup which is a q-group. Then G has a faithful irreducible module over $GF(p)$ for $p \neq q$.

Proof Take $H = 1$ in Lemma 1.14. $\square$

F. Primitive groups.

A group is called *primitive* if it has a unique self-centralizing minimal normal subgroup. Equivalently a group is primitive if it has a maximal subgroup having trivial core.

Note that, for a primitive group G, the Fitting subgroup $F(G)$ is the unique minimal normal subgroup of G. For a primitive group G we set *Stab(G)* $= \{S \lessdot G : \mathrm{Core}_G(S) = 1\}$, the set of stabilizers of G. Then if $S \in \mathrm{Stab}(G)$, it follows that S complements $F(G)$ in G. Any two elements of $\mathrm{Stab}(G)$ are conjugate in G.

We can now state another Corollary to Lemma 1.14.

(1.16) Corollary (Hawkes [5] 3.4)

Let G be a primitive group and p a prime not dividing $|F(G)|$. If H is a subgroup of G not containing $F(G)$, and U is an irreducible $GF(p).H$-module, then there exists an irreducible $GF(p).G$-module V, faithful for G and such that $U \in Q(V_H)$. □

It now follows that if G is a primitive group with p a prime not dividing $|F(G)|$, then there exists an irreducible $GF(p).G$-module faithful for G. To see that, take $H = 1$ and U the trivial $GF(p)$-module in Corollary 1.16.

(1.17) Lemma If P,Q and R are primitive groups with p,q and r distinct where they are the respective prime divisors of $|F(P)|$, $|F(Q)|$

and $|F(R)|$, then there exists an irreducible $GF(s).(P{\times}Q{\times}R)$-module faithful for $P{\times}Q{\times}R$ whenever s is a prime distinct from p,q and r .

Proof This follows immediately from Lemma 1.14 and the remark following Corollary 1.16. We take $G = P{\times}Q{\times}R$, $Soc(G) = Soc(P) \times Soc(Q) \times Soc(R)$ and $H = P{\times}1{\times}1$. □

Primitive groups provide the key to studying Schunck classes and therefore are extremely important to us . We now show how we can construct them.

(1.18) Definition

Let N be a group upon which a group G acts. We form the *semi-direct product of N with G*, written $[N]G$, which consists of elements (n,g) where $n \in N$ and $g \in G$ and which multiply according to $(n_1,g_1)(n_2,g_2) = (n_1^{g_2}n_2,g_1g_2)$. Then $[N]G$ has a normal subgroup $\bar{N} = \{(n,1_G): n \in N\}$ isomorphic with N . Also $\{(1_N,g) : g \in G\}$ is a complement to $\bar{N}$.

If V is an irreducible $GF(p).G$-module, faithful for G , then $[V]G$ is a primitive group with stabilizer isomorphic with G and V its unique minimal normal subgroup. Notice that even if V is not faithful, we still have that $[V]\, G/Ker(G \text{ on } V)$ is a primitive group since V may also be considered as a $G/Ker(G \text{ on } V)$-module.

(1.19) Definition

A group is called *polyprimitive* if all of its non-trivial quotients are primitive.

By a polyprimitive group of type $(p_1, p_2, \ldots, p_n)$, where $p_1, \ldots, p_n$ are primes satisfying $p_i \neq p_{i+1}$ for $i = 1, \ldots, n-1$, we shall mean a group G constructed in the following way.

Let G_1 be a cyclic group of order p_1. We define G_i recursively by:

Let V_i be an irreducible $GF(p_{i+1})G_i$-module, faithful for G_i. The existence of V_i follows from Corollary 1.16 since G_i has a unique minimal normal subgroup of order coprime with p_{i+1}. Set $G_{i+1} = [V_i]G_i$. Then G_{i+1} is a primitive group (in fact it is polyprimitive of type $(p_1, \ldots, p_{i+1})$ with $F(G_{i+1}) = V_i$ and $G_i \in \mathrm{Stab}(G_{i+1})$). Set $G = G_n$.

Thus S_3 is a polyprimitive group of type $(2,3)$ and S_4 is polyprimitive of type $(2,3,2)$.

The following recipe allows us to construct new primitive groups from old ones.

(1.20) Lemma

Let G be a non-cyclic primitive group and p a prime. Then

$H = G \sim Z_p$ is a primitive group with $\mathrm{Soc}(H) = \underbrace{F(G) \times \ldots\ldots \times F(G)}_{p}$.

<u>Proof</u> Lemma in Hawkes [6]. □

G. Classes of groups and closure operations.

A *class* of groups $\mathfrak{C}$ is a collection of groups with the property that if $G \in \mathfrak{C}$ and H is isomorphic with G, then $H \in \mathfrak{C}$.

An operation c, which yields a class $c\mathfrak{X}$ for each class $\mathfrak{X}$ and satisfies (i) $\mathfrak{X} \subseteq \mathfrak{Y}$ implies $c\mathfrak{X} \subseteq c\mathfrak{Y}$

(ii) $\mathfrak{X} \subseteq c\mathfrak{X} = c^2\mathfrak{X}$ for each class $\mathfrak{X}$,

is called a *closure operation*.

If $\mathfrak{X}$ is a class of groups and c is a closure operation, then $\mathfrak{X}$ is said to be c-*closed* if $c\mathfrak{X} = \mathfrak{X}$.

We describe briefly some of the closure operations which will be of use to us later. Let $\mathfrak{X}$ be a class of groups.

$Q\mathfrak{X}$ is the class of all groups which are homomorphic images of $\mathfrak{X}$-groups.

$R_0\mathfrak{X}$ is the class of groups G having normal subgroups $N_1, \ldots, N_r$ such that $\bigcap_{i=1}^{r} N_i = 1$ and $G/N_i \in \mathfrak{X}$ for $i = 1, \ldots, r$.

$E_\Phi\mathfrak{X}$ is the class consisting of all groups G for which there exists a normal subgroup N satisfying $G/N \in \mathfrak{X}$ and $N \leq \Phi(G)$. An E_Φ-closed class is called *saturated*

$S\mathfrak{X}$ is the class consisting of all subgroups of $\mathfrak{X}$-groups.

$D_0\mathfrak{X}$ is the class of all finite direct products of $\mathfrak{X}$-groups.

(1.21) Definitions

A class $\underline{\underline{F}}$ is called a *formation* if $\underline{\underline{F}}$ is both Q-closed and R_0-closed.

For an arbitrary group G, we set $G^{\underline{\underline{F}}} = \cap \{N \trianglelefteq G : G/N \in \underline{\underline{F}}\}$. Since $\underline{\underline{F}}$ is a formation, it is clear that $G^{\underline{\underline{F}}}$ is the unique smallest normal subgroup N of G such that $G/N \in \underline{\underline{F}}$. We call $G^{\underline{\underline{F}}}$ the $\underline{\underline{F}}$-*residual* of G .

(1.22) Lemma

If $\underline{\underline{F}}$ is a formation and G is a group having a normal subgroup N, then $G^{\underline{\underline{F}}}N/N$ is the $\underline{\underline{F}}$-residual of G/N .

Proof Since $G/N \,/\, G^{\underline{\underline{F}}}N/N \cong G/G^{\underline{\underline{F}}}N \in Q(G/G^{\underline{\underline{F}}}) \subseteq \underline{\underline{F}}$, if M/N is the $\underline{\underline{F}}$-residual of G/N , it follows that $G^{\underline{\underline{F}}}N \geq M \geq N$. Now $G/M \cong G/N \,/\, M/N \in \underline{\underline{F}}$ and so $M \geq G^{\underline{\underline{F}}}$. Thus $M = G^{\underline{\underline{F}}}N$. □

(1.23) Definition

If $\underline{\underline{Y}}$ is a formation and $\underline{\underline{X}}$ any class of groups, we set $\underline{\underline{X}}\,\underline{\underline{Y}} = (G : G^{\underline{\underline{Y}}} \in \underline{\underline{X}})$.

(1.24) Lemma

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are formations, then $\underline{\underline{X}}\,\underline{\underline{Y}}$ is a formation.

Proof Gaschütz [4] VII.6. □

(1.25) Definition

A Q-closed class $\underline{\underline{X}}$ is called a *Schunck class* if, whenever all primitive quotients of a group G lie in $\underline{\underline{X}}$, then G also is an $\underline{\underline{X}}$-group.

(1.26) Lemma

A formation $\underline{\underline{F}}$ is a Schunck class if and only if it is saturated.

Proof Gaschütz [4] VI.8. □

The classes $\underline{\underline{S}}_\pi$ and $\underline{\underline{N}}$ are saturated formations and hence Schunck classes.

A Schunck class is always Q- , D_0- and E_Φ-closed but need not be R_0-closed. For a set of closure operations C such that a class $\underline{\underline{X}}$ is a Schunck class if and only if it is C-closed, see Hawkes [7].

(1.27) Lemma

The unique largest formation contained in a Schunck class $\underline{\underline{X}}$ is given by

$$f(\underline{\underline{X}}) = (G \in \underline{\underline{S}} : [H/K]\ G/C_G(H/K) \in \underline{\underline{X}} \text{ for all chief factors } H/K \text{ of } G).$$

Proof Kattwinkel [9] Hilfssatz 1. □

It is easy to see that the intersection of an arbitrary collection of Schunck classes is again a Schunck class. Thus, for any class of groups $\underline{\underline{X}}$, there exists a unique minimal Schunck class containing $\underline{\underline{X}}$. We denote this by $\hat{\underline{\underline{X}}}$. Then $\hat{\underline{\underline{X}}} = (G \in \underline{\underline{S}} : Q(G) \cap \underline{\underline{P}} \subseteq Q(\underline{\underline{X}}))$ where $(G_\lambda : \lambda \in \Lambda)$ denotes the class of groups consisting of those groups isomorphic with an element of the set $\{G_\lambda : \lambda \in \Lambda\}$.

(1.28) Definition

A class of groups $\underline{\underline{X}}$ is called a D-*class* if each group G has a unique conjugacy class of maximal $\underline{\underline{X}}$-subgroups.

H. Sylow subgroups.

We denote the class of a Sylow p-subgroup of a group G by $\gamma_p(G)$.

(1.29) Lemma

Let G be a group having a primitive quotient isomorphic with a group T whose minimal normal subgroup is a p-group. If $O_p(G) = 1$, then there is a prime q distinct from p for which $\gamma_q(G) > \gamma_q(T)$.

Proof Let $\underline{\underline{F}}$ be the class defined by:

$$(1.\alpha) \qquad \underline{\underline{F}} = (G \in \underline{\underline{S}} : \mathrm{Aut}_G(H/K) \not\geq T/F(T) \text{ for all p-chief factors } H/K \text{ of } G) .$$

Certainly $\underline{\underline{F}}$ is Q-closed. It follows, from the Jordan-Hölder theorem for operator composition series, that $\underline{\underline{F}}$ is R_0-closed and hence is a formation. Furthermore $\underline{\underline{F}} = \underline{\underline{S}}_{p'}\underline{\underline{F}}$.

Let $F(G) = F_1 \times \ldots\ldots \times F_n$ where F_i is a Sylow q_i-group of $F(G)$ for $i \in \{1,\ldots\ldots,n\}$. Then $\bigcap_i C_G(F_i) = C_G(F(G)) \leq F(G)$ by Lemma 1.2. Since $O_p(G) = 1$, we have $p \nmid |\bigcap_i C_G(F_i)|$. Suppose $G/C_G(F_i) \in \underline{\underline{F}}$ for all $i \in \{1,\ldots,n\}$. Then $G/\bigcap_i C_G(F_i) \in \underline{\underline{F}}$ and so $G \in \underline{\underline{S}}_{p'}\underline{\underline{F}} = \underline{\underline{F}}$. This contradiction shows there is an i for which $G/C_G(F_i) \notin \underline{\underline{F}}$.

Let $Y \in \mathrm{Syl}_{q_i}(G)$. Since $F_i \leq Y$, clearly $Z(Y)$ centralizes F_i and hence $Z(Y) \leq C_G(F_i)$. Therefore $\gamma_{q_i}(G/C_G(F_i)) < \gamma_{q_i}(G)$.

Now $G/C_G(F_i) \notin \underline{\underline{F}}$ implies that $\gamma_{q_i}(G/C_G(F_i)) \geq \gamma_{q_i}(T)$. Hence $\gamma_{q_i}(G) > \gamma_{q_i}(T)$. □

(1.30) <u>Definition</u>

Let G be a group with π the set of prime divisors of $|G|$. For each subset σ of π , we set $S^\sigma = \bigcap_{p \epsilon \sigma} S^p$ where S^p is a Sylow p-complement of G . Then $\mathcal{S}(G) = \{S^{p'} : p \in \pi\}$ is called a *Sylow system* for G and consists of Sylow subgroups of G .

(1.31) <u>Lemma</u>

If H is a subgroup of G with a Sylow system $\mathcal{S}(H)$, there is a Sylow system $\mathcal{S}(G)$ of G such that $\mathcal{S}(H) = \{T \cap H : T \in \mathcal{S}(G)\}$. Here $\mathcal{S}(G)$ is said to *reduce* into H .

<u>Proof</u> Huppert [8] VI.2.5. □

§2. Schunck classes and $\underline{D}$-classes.

We recall that a Schunck class $\underline{\underline{X}}$ is a Q-closed class of groups with the property that, if all primitive quotients of a group G lie in $\underline{\underline{X}}$, then G is itself an $\underline{\underline{X}}$-group. Theorem 2.2 will provide us with a more useful characterization of Schunck classes. First we need a definition.

(2.1) Definition

Let $\underline{\underline{X}}$ be a class of groups. An $\underline{\underline{X}}$-subgroup H of G is called an $\underline{\underline{X}}$-*projector* of G if H is a maximal $\underline{\underline{X}}$-subgroup (that is, H is not contained in any larger $\underline{\underline{X}}$-subgroup of G), and furthermore, for any epimorphism ϕ of G the image of H under ϕ is a maximal $\underline{\underline{X}}$-subgroup of the image of G.

(2.2) Theorem (Schunck)

A class of groups $\underline{\underline{X}}$ is a Schunck class if and only if every group has an $\underline{\underline{X}}$-projector. Furthermore for any group the $\underline{\underline{X}}$-projectors are conjugate if $\underline{\underline{X}}$ is a Schunck class, and are denoted by $\mathrm{Proj}_{\underline{\underline{X}}}(G)$ for a group G.

Proof Schunck [10] Theorem 4.4. □

(2.3) Example

Let p be a prime and $\underline{\underline{S}}_p$ the class of all finite soluble p-groups.

Then Sylow's theorem shows that $\underline{\underline{S}}_p$ is a Schunck class with the $\underline{\underline{S}}_p$-projectors corresponding to the Sylow p-subgroups. □

(2.4) Example

Let $\underline{\underline{N}}$ be the class of nilpotent groups. Then $\underline{\underline{N}}$ is a Schunck class, the $\underline{\underline{N}}$-projectors being the Carter subgroups. (See Huppert [8] VI.12.)

We know from Sylow theory that for any group every $\underline{\underline{S}}_p$-subgroup is contained in an $\underline{\underline{S}}_p$-projector. However it is not true that for every group all its $\underline{\underline{N}}$-subgroups are contained in $\underline{\underline{N}}$-projectors. For example in S_3 the Sylow 3-subgroup is nilpotent but all $\underline{\underline{N}}$-projectors have order 2.

(2.5) Definition

A Schunck class is called a *Schunck* D-*class* or a $\underline{D}$-*class* if it is also a D-class.

Therefore $\underline{\underline{S}}_p$ is a $\underline{D}$-class for each prime p but $\underline{\underline{N}}$ is not.

A very useful concept in the theory of Schunck classes is that of a Schunck boundary as introduced by Doerk.

(2.6) Definition

A class of groups $\underline{\underline{B}}$ is called a *boundary* if for each $B \in \underline{\underline{B}}$, we have $Q(B) \cap \underline{\underline{B}} = (B)$.

If all groups in a boundary $\underline{\underline{B}}$ are primitive, then $\underline{\underline{B}}$ is called a *Schunck boundary*.

(2.7) Definition

Let $\underline{\underline{B}}$ be a class of groups. We set

$$h(\underline{\underline{B}}) = (G \in \underline{\underline{S}} : Q(G) \cap \underline{\underline{B}} \subseteq \underline{\underline{I}}) .$$

Any group in $h(\underline{\underline{B}})$ is said to be $\underline{\underline{B}}$-*perfect*. Notice that $h(\underline{\underline{B}})$ is always Q-closed.

(2.8) Lemma

If $\underline{\underline{B}}$ is a Schunck boundary, then $h(\underline{\underline{B}})$ is a Schunck class.

Proof Let G be a group having all its primitive quotients in $h(\underline{\underline{B}})$. Then, since all $\underline{\underline{B}}$-groups are primitive, $Q(G) \cap \underline{\underline{B}} = \phi$ and so $G \in h(\underline{\underline{B}})$. □

(2.9) Lemma (Doerk)

Let $\underline{\underline{X}}$ be a Schunck class. Then the class $b(\underline{\underline{X}}) = (G \notin \underline{\underline{X}} : (Q-1)(G) \subseteq \underline{\underline{X}})$ is a Schunck boundary and $\underline{\underline{X}} = h(b(\underline{\underline{X}}))$.

Proof It is easy to see that $b(\underline{\underline{X}})$ is a boundary since $\underline{\underline{X}}$ is a Q-closed class. If G is a non-primitive group and $(Q-1)(G) \subseteq \underline{\underline{X}}$, then, by the definition of a Schunck class, we have $G \in \underline{\underline{X}}$. Therefore every $b(\underline{\underline{X}})$-group must be primitive and so $b(\underline{\underline{X}})$ is a Schunck boundary.

Since $\underline{\underline{X}}$ is a Q-closed class, no $\underline{\underline{X}}$-group can have a quotient in $b(\underline{\underline{X}})$ and so $\underline{\underline{X}} \subseteq h(b(\underline{\underline{X}}))$. Suppose that $h(b(\underline{\underline{X}})) \neq \underline{\underline{X}}$ and choose a group G with minimal order in $h(b(\underline{\underline{X}}))\backslash\underline{\underline{X}}$. Since $G \in h(b(\underline{\underline{X}}))$, it follows that $(Q-1)(G) \subseteq h(b(\underline{\underline{X}}))$ hence $(Q-1)(G) \subseteq \underline{\underline{X}}$ by choice of G . Then $G \in b(\underline{\underline{X}})$ by definition of $b(\underline{\underline{X}})$. This contradicts $G \in h(b(\underline{\underline{X}}))$ and so we must have $\underline{\underline{X}} = h(b(\underline{\underline{X}}))$. □

The following result is one we shall use frequently, and without reference.

(2.10) Lemma

If $\underline{\underline{X}}$ is a Schunck class and $X \in \mathrm{Proj}_{\underline{\underline{X}}}(G)$ for a group G , then $X \in \mathrm{Proj}_{\underline{\underline{X}}}(J)$ for each subgroup J of G containing X .

Proof Gaschütz [4] II.15. □

(2.11) Definition

Let $\underline{\underline{X}}$ be a Schunck class and G a group. A $b(\underline{\underline{X}})$-group B is called a *cast-off group* for G if there is a subgroup of G containing an $\underline{\underline{X}}$-projector of G and having a quotient isomorphic with B .

A subclass $\underline{\underline{B}}$ of $b(\underline{\underline{X}})$ is called an $\underline{\underline{X}}$-*cast-off class* for G if $\mathrm{Proj}_{h(\underline{\underline{B}})}(G) = \mathrm{Proj}_{\underline{\underline{X}}}(G)$.

(2.12) Definition

Let G be a primitive group and let $K \in \mathrm{Stab}(G)$. Let J be a subgroup of K and let V be a J-composition factor of $F(G)$. Then the semi-direct product $H = [V]J/C_J(V)$ is called a (G,J)-*pass group*. (Primitive Action of a Subgroup on the Socle)

A group H is called a G-*pass group* if it is isomorphic with some (G,J)-pass group as above. The class of all G-pass groups is written $p(G)$.

(2.13) Lemma

If H is a G-pass group and L is an H-pass group, then L is a G-pass group.

Proof Let $K \in \mathrm{Stab}(G)$ and choose a subgroup J of K such that H is a (G,J)-pass group. Thus $H \cong [V]J/C_J(V)$ for some J-composition factor V of $F(G)$. Let $S \in \mathrm{Stab}(H)$ and let T be a subgroup of S such that L is an (H,T)-pass group. Then $L \cong [U]T/C_T(U)$ for some T-composition factor of $F(H) = V$. Now S is isomorphic with $J/C_J(V)$ and so there is a subgroup R of K containing $C_J(V)$ such that $R/C_J(V)$ is isomorphic with T and U is an R-composition factor of V, and hence of $F(G)$. Thus L is isomorphic with $[U]R/C_R(U)$ and so is a G-pass group. □

(2.14) Lemma

Let B and C be non-isomorphic groups such that (B,C) is a Schunck boundary and $C \in Q(G)$ for some group G. Let X be a subgroup of G containing an $h(B)$-projector of G. Then $C \in Q(X)$.

Proof Let M/N be a chief factor of G such that X/N complements M/N in G/N. Then $G/N \cong B$. Let $T \trianglelefteq G$ be such that $G/T \cong C$. Then T must cover M/N since (B,C) is a Schunck boundary. Therefore $G = XM = XNT = XT$ and so $C \cong G/T = XT/T \cong X/X \cap T$ and the result holds. $\square$

(2.15) Lemma

Let $\underline{\underline{X}}$ be a Schunck class and G a group. Then G has a unique smallest $\underline{\underline{X}}$-cast-off class.

Proof We suppose the conclusion to be false and take G as a counterexample. If $\underline{\underline{B}}$ is an $\underline{\underline{X}}$-cast-off class, then $(B \in \underline{\underline{B}} : |B| \leq |G|)$ is a cast-off class for G. It is now plain that G has minimal cast-off classes. Let $\underline{\underline{B}}$ and $\underline{\underline{C}}$ be two such. Since $\underline{\underline{B}} \cap \underline{\underline{C}}$ is not a cast-off class, if $Y \in \mathrm{Proj}_{h(\underline{\underline{B}}\cap\underline{\underline{C}})}(G)$ and X is an $\underline{\underline{X}}$-projector of G contained in Y, then $Y \neq X$. Since $\underline{\underline{B}}$ is a cast-off class for G, we have $X \in \mathrm{Proj}_{h(\underline{\underline{B}})}(G)$. Lemma 2.10 yields $X \in \mathrm{Proj}_{h(\underline{\underline{B}})}(Y)$ and $X \in \mathrm{Proj}_{h(\underline{\underline{C}})}(Y)$. Let B be a $(\underline{\underline{B}}\backslash\underline{\underline{B}}\cap\underline{\underline{C}})$-quotient of Y. By repeated application of Lemma 2.14, we see that an $h(\underline{\underline{C}})$-projector of Y has a quotient isomorphic with B. This contradicts $X \in \mathrm{Proj}_{h(\underline{\underline{C}})}(Y)$, $X \in \underline{\underline{X}}$ and $B \in b(\underline{\underline{X}})$, and the proof is complete. $\square$

Associated with any Schunck class $\underline{\underline{X}}$ is a set of primes, $\mathrm{char}(\underline{\underline{X}})$, consisting of those primes p for which Z_p is an $\underline{\underline{X}}$-group. This set is called the *characteristic* of $\underline{\underline{X}}$. Equivalently, $\mathrm{char}(\underline{\underline{X}})$ is the set of primes p for which Z_p is not a $b(\underline{\underline{X}})$-group. For example, $\mathrm{char}(\underline{\underline{S}}_\pi) = \pi$ and $\mathrm{char}(\underline{\underline{N}}) = \mathbf{P}$.

B. Strong Containment.

(2.16) Definition

Let $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ be Schunck classes with the property that, for all soluble groups G, every $\underline{\underline{X}}$-projector of G is contained in some $\underline{\underline{Y}}$-projector of G. Then we say that $\underline{\underline{X}}$ is strongly contained in $\underline{\underline{Y}}$ (or $\underline{\underline{Y}}$ strongly contains $\underline{\underline{X}}$) and write $\underline{\underline{X}} << \underline{\underline{Y}}$ or $\underline{\underline{Y}} >> \underline{\underline{X}}$.

(2.17) Example

Let σ,π be sets of primes with $\sigma \subset \pi$. Then $\underline{\underline{S}}_\sigma << \underline{\underline{S}}_\pi$. This follows from the properties of Hall subgroups since the $\underline{\underline{S}}_\sigma$ ($\underline{\underline{S}}_\pi$)-projectors are just the Hall σ (π)-subgroups.

In [1] Doerk introduces the following concept which plays an important role in most problems involving strong containment properties.

(2.18) Definition

Let $\underline{\underline{X}}$ be a Schunck class. The *avoidance class of* $\underline{\underline{X}}$, written $a(\underline{\underline{X}})$, is the class given by:

$$a(\underline{\underline{X}}) = (G \in \underline{\underline{P}} : \text{If } X \in \mathrm{Proj}_{\underline{\underline{X}}}(G) \text{, then } X \cap F(G) = 1.) .$$

Notice that $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{X}})$. The following result due to Doerk shows the significance of the avoidance class:

(2.19) Theorem

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are Schunck classes, the following are equivalent:

(i) $\underline{\underline{X}} \gg \underline{\underline{Y}}$

(ii) $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{Y}})$

(iii) $a(\underline{\underline{X}}) \subseteq a(\underline{\underline{Y}})$

Proof Doerk [1] 2.2. □

When considering problems concerning strong containment, it will often be easier to consider avoidance classes associated with the Schunck classes rather than the Schunck classes themselves.

In [4] Gaschütz attributes to Blessenohl a method for defining the composite of two Schunck classes to yield a new Schunck class. We take Hawkes version of this definition.

(2.20) Definition

If $\{\underline{\underline{H}}_\lambda : \lambda \in \Lambda\}$ is a set of Schunck classes, we define their *composite* $\langle\underline{\underline{H}}_\lambda : \lambda \in \Lambda\rangle$ to be

$$(G : G = \langle Proj(\Sigma,\underline{\underline{H}}_\lambda) : \lambda \in \Lambda\rangle) \quad ,$$

where Σ is a fixed Sylow system of G and $Proj(\Sigma,\underline{\underline{H}})$ is the unique $\underline{\underline{H}}$-projector of G into which Σ reduces.

(2.21) Lemma

If $\{\underline{\underline{H}}_\lambda : \lambda \in \Lambda\}$ is a set of Schunck classes, then $\langle \underline{\underline{H}}_\lambda : \lambda \in \Lambda \rangle$ is a Schunck class. Furthermore, for a group G with a Sylow system Σ, $\langle \mathrm{Proj}(\Sigma, \underline{\underline{H}}_\lambda) : \lambda \in \Lambda \rangle$ is an $\langle \underline{\underline{H}}_\lambda : \lambda \in \Lambda \rangle$-projector of G.

Proof Hawkes [5], 1.5. □

We can now make the family of Schunck classes into a lattice, denoted by $\underline{H}$, with meet and join operations given by

$$\bigvee_{\lambda \in \Lambda} \underline{\underline{X}}_\lambda = \langle \underline{\underline{X}}_\lambda : \lambda \in \Lambda \rangle$$

$$\begin{aligned} \bigwedge_{\lambda \in \Lambda} \underline{\underline{X}}_\lambda &= \sup\{\underline{\underline{L}} \in \underline{H} : \underline{\underline{L}} \ll \underline{\underline{X}}_\lambda \text{ for all } \lambda \in \Lambda\} \\ &= \langle \underline{\underline{L}} : \underline{\underline{L}} \ll \underline{\underline{X}}_\lambda \text{ for all } \lambda \in \Lambda \rangle \end{aligned}$$

Notice that for two Schunck classes $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ we have $\underline{\underline{X}} \wedge \underline{\underline{Y}} \subseteq \underline{\underline{X}} \cap \underline{\underline{Y}}$ but in general we do not have equality. However, in the case where both $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes, we do have $\underline{\underline{X}} \wedge \underline{\underline{Y}} = \underline{\underline{X}} \cap \underline{\underline{Y}}$, and Wood [11] shows that $\underline{\underline{X}} \cap \underline{\underline{Y}}$ is again a $\underline{D}$-class. He also shows that $\langle \underline{\underline{X}}, \underline{\underline{Y}} \rangle$ is a $\underline{D}$-class. Thus the family of all $\underline{D}$-classes forms a sublattice of $\underline{H}$ and we denote this by $\underline{D}$.

The following results will be of considerable use in describing $\vee$ and $\wedge$:

(2.22) Lemma

For a family of Schunck classes $\{\underline{\underline{X}}_\lambda : \lambda \in \Lambda\}$, we have:

(i) $a\left(\bigvee_{\lambda\in\Lambda} \underline{\underline{X}}_\lambda\right) = \bigcap_{\lambda\in\Lambda} a(\underline{\underline{X}}_\lambda)$

(ii) $a\left(\bigwedge_{\lambda\in\Lambda} \underline{\underline{X}}_\lambda\right) = \bigcap_{\underline{\underline{Z}}\in\underline{Z}} a(\underline{\underline{Z}})$, where $\underline{Z} = \{\underline{\underline{Z}}\in\underline{\underline{H}}:\underline{\underline{Z}} << \underline{\underline{X}}_\lambda \text{ for all } \lambda \in \Lambda\}$.

Proof (i) Let $\underline{\underline{U}} = \bigvee_{\lambda\in\Lambda} \underline{\underline{X}}_\lambda$. Then $\underline{\underline{U}} >> \underline{\underline{X}}_\lambda$ for each λ and so $a(\underline{\underline{U}}) \subseteq \bigcap_{\lambda\in\Lambda} a(\underline{\underline{X}}_\lambda)$ by Theorem 2.19. Let G be an $(\bigcap_{\lambda\in\Lambda} a(\underline{\underline{X}}_\lambda))$-group, if one exists, and let $K \in \mathrm{Stab}(G)$. Then $\mathrm{Proj}_{\underline{\underline{X}}_\lambda}(K) \subseteq \mathrm{Proj}_{\underline{\underline{X}}_\lambda}(G)$ for each λ . Let $X_\lambda \in \mathrm{Proj}_{\underline{\underline{X}}_\lambda}(K)$ for each $\lambda \in \Lambda$ be chosen such that $\langle X_\lambda : \lambda \in \Lambda\rangle \in \mathrm{Proj}_{\underline{\underline{U}}}(K)$. Then clearly $\langle X_\lambda : \lambda \in \Lambda\rangle \in \mathrm{Proj}_{\underline{\underline{U}}}(G)$ and so $G \in a(\underline{\underline{U}})$.

(ii) This follows from (i) and the definition of $\underline{\underline{X}} \wedge \underline{\underline{Y}}$. □

Next we define classes closely associated with $a(\underline{\underline{X}})$ and $b(\underline{\underline{X}})$ for a Schunck class $\underline{\underline{X}}$. Let π be any set of primes. We set

$b_\pi(\underline{\underline{X}}) = (B \in b(\underline{\underline{X}}) : F(B) \text{ is a } \pi\text{-group})$

$a_\pi(\underline{\underline{X}}) = (A \in a(\underline{\underline{X}}) : F(A) \text{ is a } \pi\text{-group})$

$c(\underline{\underline{X}}) = (B/F(B) : B \in b(\underline{\underline{X}}))$

$c_\pi(\underline{\underline{X}}) = (B/F(B) : B \in b_\pi(\underline{\underline{X}}))$.

(2.23) Lemma (Doerk [1] 2.6)

Let $\underline{\underline{X}}$ be a Schunck class and suppose that G is an $(a(\underline{\underline{X}})\backslash b(\underline{\underline{X}}))$-group. Let $K \in \mathrm{Stab}(G)$, $X \leq \mathrm{Proj}_{\underline{\underline{X}}}(K) \leq \mathrm{Proj}_{\underline{\underline{X}}}(G)$ and let Y be a subgroup of K containing X . Then all (G,Y)-pass groups lie in $a(\underline{\underline{X}})$, and if $Y = X$, then all lie in $b(\underline{\underline{X}})$.

Proof Let A be a (G,Y)-pass group. Then A has form $[R/S]Y/C_Y(R/S)$ for some Y-composition factor R/S of $F(G)$. Then A is a primitive group. Let $\bar{A} = RY/M$ where $M = C_{YS}(R/S)$. Then $A \cong \bar{A}$ and $YM/M \in \mathrm{Stab}(\bar{A})$. Now the definition of a projector and Lemma 2.10 combine to give $X \in \mathrm{Proj}_{\underline{\underline{X}}}(RY)$, and hence $XM/M \in \mathrm{Proj}_{\underline{\underline{X}}}(\bar{A})$. Since $XM/M \leq YM/M$, it follows that $\bar{A} \in a(\underline{\underline{X}})$.

If $Y = X$, then $A/F(A) \in Q(X) \subseteq \underline{\underline{X}}$ and so $A \in b(\underline{\underline{X}})$. □

It is now clear that if $a_p(\underline{\underline{X}}) \neq \phi$ for a Schunck class $\underline{\underline{X}}$, then $b_p(\underline{\underline{X}}) \neq \phi$.

(2.24) Lemma

Let G be a primitive group and $\underline{\underline{X}}$ a Schunck class. Suppose that, for $K \in \mathrm{Stab}(G)$ and $X \in \mathrm{Proj}_{\underline{\underline{X}}}(K)$, all (G,X)-pass groups lie in $a(\underline{\underline{X}})$ (and hence in $b(\underline{\underline{X}})$) . Then G is an $a(\underline{\underline{X}})$-group.

Proof Let V be a maximal X-submodule of $F(G)$. Setting

$M = C_{VX}(F(G)/V)$, we have $F(G)X/M \cong [F(G)/V]X/C_X(F(G)/V)$ which is a $b(\underline{\underline{X}})$-group by hypothesis. Therefore $X \in \mathrm{Proj}_{\underline{\underline{X}}}([F(G)/V]X)$. Now take U to be a maximal X-submodule of V , then we see that $X \in \mathrm{Proj}_{\underline{\underline{X}}}([F(G)/U]X)$. Repeating as necessary yields $X \in \mathrm{Proj}_{\underline{\underline{X}}}(F(G).X)$. Therefore $X \in \mathrm{Proj}_{\underline{\underline{X}}}(G)$ and $G \in a(\underline{\underline{X}})$. □

(2.25) Lemma

Let G be a primitive group and $\underline{\underline{X}}$ a Schunck class. Let $K \in \mathrm{Stab}(G)$ and $X \in \mathrm{Proj}_{\underline{\underline{X}}}(K)$. If there exists a subgroup Y of K containing X such that all (G,Y)-pass groups lie in $a(\underline{\underline{X}})$, then G is an $a(\underline{\underline{X}})$-group.

Proof Let T be a (G,Y)-pass group. Hence $T = [V]Y/C_Y(V)$ for some Y-composition factor of $F(G)$. By Lemma 2.23 all $(T,XC_Y(V)/C_Y(V))$-pass groups lie in $b(\underline{\underline{X}})$ and hence all (G,X)-pass groups lie in $b(\underline{\underline{X}})$. Therefore $G \in a(\underline{\underline{X}})$ by Lemma 2.24. □

The following result will be used many times later on.

(2.26) Lemma (Förster [2] 2.9)

Let $\underline{\underline{X}}$ be a Schunck class with G an $a(\underline{\underline{X}})$-group. Let $K \in \mathrm{Stab}(G)$ and $X \in \mathrm{Proj}_{\underline{\underline{X}}}(K) \subseteq \mathrm{Proj}_{\underline{\underline{X}}}(G)$. If p is the prime divisor of $|F(G)|$, then $X/O_p(X) \in R_0\, c_p(\underline{\underline{X}})$ and hence $X \in \underline{\underline{S}}_p\, Q\, R_0\, c_p(\underline{\underline{X}})$.

<u>Proof</u> By Lemma 2.23 all (G,X)-pass groups lie in $b(\underline{\underline{X}})$ and hence in $b_p(\underline{\underline{X}})$ since p is the only prime dividing $|F(G)|$. Let $1 < Y_1 < \ldots\ldots < Y_r = F(G)$ be an X-composition series for $F(G)$ and set $V_i = Y_i/Y_{i-1}$. Then we have $[V_i]X/C_X(V_i) \in b_p(\underline{\underline{X}})$ for $i = 1,\ldots,r$. Hence $X/C_X(V_i) \in c_p(\underline{\underline{X}})$ for $i = 1,\ldots,r$. Since $F(G)$ is faithful as an X-module, we have $\mathrm{Ker}(X \text{ on } F(G)) = O_p(X)$ by Lemma 1.10. Therefore $\bigcap_{i=1}^{r} C_X(V_i = O_p(X)$ and hence $X/O_p(X) \in R_0\, c_p(\underline{\underline{X}})$. □

(2.27) <u>Corollary</u>

If $\underline{\underline{X}}$ is a Schunck class satisfying $b(\underline{\underline{X}}) = b_p(\underline{\underline{X}})$ for some prime p, and G,K and X are as in Lemma 2.26, then $O_p(X) = 1$ and so $X \in R_0\, c_p(\underline{\underline{X}})$.

<u>Proof</u> By Lemma 2.26 it is enough to show that $O_p(X) = 1$. Since $G \in a(\underline{\underline{X}})$ and $b(\underline{\underline{X}}) = b_p(\underline{\underline{X}})$, it follows that $F(G)$ is a p-group. Furthermore $F(G)$ is a faithful and irreducible K-module. Therefore $O_p(K) = 1$ by Lemma 1.10. Now $b(\underline{\underline{X}}) = b_p(\underline{\underline{X}})$ implies that $F(K) \leq X \leq K$ and hence $F(K) \leq F(X)$. Now $Z(O_p(X))$ centralizes $F(X)$ and hence $F(K)$. By Lemma 1.2 we have $F(K) \geq C_K(F(K)) \geq Z(O_p(X))$. Thus $Z(O_p(X)) \leq O_p(K) = 1$. Since $O_p(X)$ is a p-group, it must have non-trivial centre if it is itself non-trivial. Hence $O_p(X) = 1$, as required. □

Schunck classes $\underline{\underline{X}}$ having the property $b(\underline{\underline{X}}) = b_p(\underline{\underline{X}})$ will be important throughout this thesis. Notice that given such a class and any group G an $\underline{\underline{X}}$-projector of G contains a Sylow q-subgroup of G for each prime q distinct from p .

C. Sublattices of $\underline{D}$.

We describe briefly two sublattices of $\underline{D}$.

(2.28) Definition

A group G is said to be *π-perfect* if $O^{\pi}(G) = G$ for a set of primes π . The class of all π-perfect groups, written $\underline{\underline{Q}}^{\pi}$, is a Schunck class and $b(\underline{\underline{Q}}^{\pi}) = (Z_p : p \in \pi)$.

It is easy to see that $\underline{\underline{Q}}^{\pi} \wedge \underline{\underline{Q}}^{\sigma} = \underline{\underline{Q}}^{\pi \cup \sigma}$ and $\underline{\underline{Q}}^{\pi} \vee \underline{\underline{Q}}^{\sigma} = \underline{\underline{Q}}^{\pi \cap \sigma}$ for any two sets of primes π and σ . Thus $\underline{L} = \{\underline{\underline{Q}}^{\pi} : \pi \subseteq \mathbb{P}\}$ is a sublattice of $\underline{D}$.

The second sublattice consists of all classes $\underline{\underline{S}}_{\pi}$ of π-groups where π is any set of primes. Here we have $\underline{\underline{S}}_{\pi} \wedge \underline{\underline{S}}_{\sigma} = \underline{\underline{S}}_{\pi \cap \sigma}$ and $\underline{\underline{S}}_{\pi} \vee \underline{\underline{S}}_{\sigma} = \underline{\underline{S}}_{\pi \cup \sigma}$ for sets of primes π and σ .

D. Maximality in the Schunck class lattice.

(2.29) Definitions

A Schunck class $\underline{\underline{X}}$ is said to be *maximal* in a Schunck class $\underline{\underline{Y}}$ if $\underline{\underline{X}} << \underline{\underline{Y}}$ and there is no Schunck class which is properly strongly contained in $\underline{\underline{Y}}$ and properly strongly contains $\underline{\underline{X}}$. A Schunck class is called *maximal* or 1-*maximal* if it is maximal in $\underline{\underline{S}}$.

Two ideas of n-maximality have been studied in the Schunck class lattice. Doerk considered n-step maximality where a Schunck class $\underline{\underline{X}}$ is n-step maximal if and only if it is maximal in an (n-1)-step maximal Schunck class. However there are Schunck classes which are 2-step maximal but which have proper chains to $\underline{\underline{S}}$ with arbitrarily large length. (See example 4.5.) We will concern ourselves with Schunck classes which are n-maximal according to the following definition.

(2.30) Definition

A Schunck class $\underline{\underline{H}}$ is said to be n-maximal in $\underline{H}$ if any proper chain of Schunck classes from $\underline{\underline{H}}$ to $\underline{\underline{S}}$ can be obtained by removing links in a proper chain from $\underline{\underline{H}}$ to $\underline{\underline{S}}$ of length n.

The (1-)maximal Schunck classes were characterized by Doerk as in Theorem 2.35 and later Förster characterized the 2-maximal Schunck classes.

(2.31) Lemma

If $\underline{\underline{H}}$ is an n-maximal Schunck class, then $|b(\underline{\underline{H}})| \leq n$.

Proof Suppose $|b(\underline{\underline{H}})| > n$. Let $(B_1,\ldots,B_n) \subset b(\underline{\underline{H}})$ with $B_1,\ldots,B_n$ pairwise non-isomorphic and set $\underline{\underline{B}}_i = (B_1,\ldots,B_i)$ for $i = 1,\ldots,n$. Then let $\underline{\underline{X}}_i$ be the Schunck class corresponding to the Schunck boundary $\underline{\underline{B}}_i$. Since $\underline{\underline{B}}_i \subseteq \underline{\underline{B}}_{i+1}$, it follows that $\underline{\underline{X}}_i >> \underline{\underline{X}}_{i+1}$ by Theorem 2.19 and also $\underline{\underline{X}}_n >> \underline{\underline{H}}$. Then $\underline{\underline{S}} >> \underline{\underline{X}}_1 >> \ldots\ldots >> \underline{\underline{X}}_n >> \underline{\underline{H}}$ is a proper chain of Schunck classes from $\underline{\underline{H}}$ to $\underline{\underline{S}}$ of length n+1 . This contradicts the n-maximality of $\underline{\underline{H}}$ and so $|b(\underline{\underline{H}})| \leq n$. □

(2.32) Lemma

If $\underline{\underline{H}}$ and $\underline{\underline{X}}$ are Schunck classes satisfying $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$ and $\underline{\underline{X}} >> \underline{\underline{H}}$, then $a(\underline{\underline{X}}) = b(\underline{\underline{X}})$.

Proof Since $\underline{\underline{X}} >> \underline{\underline{H}}$, we have $a(\underline{\underline{X}}) \subseteq a(\underline{\underline{H}}) = b(\underline{\underline{H}})$ by Theorem 2.19. Suppose $G \in a(\underline{\underline{X}}) \backslash b(\underline{\underline{X}})$. Then $G \in b(\underline{\underline{H}})$ and G has a quotient isomorphic with some $b(\underline{\underline{X}})$-group, and hence a $b(\underline{\underline{H}})$-group, T say. But $b(\underline{\underline{H}})$ is a boundary and so we cannot have both G and T in $b(\underline{\underline{H}})$. Therefore no such G exists and $a(\underline{\underline{X}}) = b(\underline{\underline{X}})$. □

(2.33) Lemma

If $\underline{\underline{H}}$ is a Schunck class with $|a(\underline{\underline{H}})| = |b(\underline{\underline{H}})| = n$, then $\underline{\underline{H}}$ is n-maximal in $\underline{H}$.

<u>Proof</u> Since $|a(\underline{\underline{H}})|$ is finite and to have $\underline{\underline{A}} \lneq \underline{\underline{B}}$ for Schunck classes $\underline{\underline{A}}$ and $\underline{\underline{B}}$, we need $a(\underline{\underline{A}}) \supsetneq a(\underline{\underline{B}})$, it follows that all proper chains from $\underline{\underline{H}}$ to $\underline{\underline{S}}$ have length at most n. Let $\underline{\underline{H}} = \underline{\underline{H}}_r << \underline{\underline{H}}_{r-1} << \ldots << \underline{\underline{H}}_1 << \underline{\underline{H}}_0 = S$ be a maximally refined chain, C, from $\underline{\underline{H}}$ to $\underline{\underline{S}}$. Suppose $r < n$ and choose the smallest i for which $|b(\underline{\underline{H}}_i)| > i$. Then $b(\underline{\underline{H}}_{i-1}) \subseteq a(\underline{\underline{H}}_i) = b(\underline{\underline{H}}_i)$ by Theorem 2.19 and Lemma 2.32. By choice of i, we have $|b(\underline{\underline{H}}_i)\backslash b(\underline{\underline{H}}_{i-1})| > 1$. Let B be any $(b(\underline{\underline{H}}_i)\backslash b(\underline{\underline{H}}_{i-1}))$-group and set $b(\underline{\underline{X}}) = b(\underline{\underline{H}}_{i-1}) \cup (B)$. Then $\underline{\underline{H}}_{i-1} \underset{\neq}{>>} \underline{\underline{X}} \underset{\neq}{>>} \underline{\underline{H}}_i$ and thus C is not maximally refined. Therefore $r = n$ and so $\underline{\underline{H}}$ is n-maximal in $\underline{\underline{H}}$. □

We are now in a position to begin our characterizations of maximal and 2-maximal Schunck classes.

(2.34) <u>Lemma</u>

If $\underline{\underline{H}}$ is a Schunck class with $|b(\underline{\underline{H}})| = 1$, then $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$.

<u>Proof</u> Let $b(\underline{\underline{H}}) = (H)$ and let p be the prime divisor of $|F(H)|$. We suppose $a(\underline{\underline{H}}) \neq b(\underline{\underline{H}})$ and choose G in $(a(\underline{\underline{H}})\backslash b(\underline{\underline{H}}))$. Then G has a quotient isomorphic with H and $p \,\big|\, |F(G)|$. Applying Lemma 1.29 to $K \in \mathrm{Stab}(G)$ gives $\gamma_q(G) = \gamma_q(K) > \gamma_q(H)$ for some prime $q \neq p$. Then if $X \in \mathrm{Proj}_{\underline{\underline{H}}}(G)$, we have $\gamma_q(X) = \gamma_q(G)$ since X contains a Sylow q-subgroup of G. However by Corollary 2.27 we have $X \in R_0\, c(\underline{\underline{H}}) \subseteq$ the formation of all finite soluble groups having Sylow q-subgroups of

class at most equal to $\gamma_q(H)$. This contradiction yields $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$. $\square$

(2.35) Theorem (Doerk [1] 2.4)

A Schunck class $\underline{\underline{H}}$ is maximal in $\underline{H}$ if and only if $|b(\underline{\underline{H}})| = 1$.

Proof The necessity of $|b(\underline{\underline{H}})| = 1$ follows from Lemma 2.31.

We suppose now that $|b(\underline{\underline{H}})| = 1$. Then by Lemma 2.34 we have $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$. If $\underline{\underline{X}} >> \underline{\underline{H}}$, then $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{H}}) = b(\underline{\underline{H}})$ by Theorem 2.19 and so either $\underline{\underline{H}} = \underline{\underline{X}}$ or $\underline{\underline{H}} = \underline{\underline{S}}$ and therefore $\underline{\underline{H}}$ is maximal. $\square$

(2.36) Theorem (Förster [2] 7.6)

A Schunck class $\underline{\underline{H}}$ is 2-maximal in $\underline{H}$ if and only if $|a(\underline{\underline{H}})| = |b(\underline{\underline{H}})| = 2$.

Proof The sufficiency of $|a(\underline{\underline{H}})| = |b(\underline{\underline{H}})| = 2$ is dealt with in Lemma 2.33.

We suppose that $\underline{\underline{H}}$ is a 2-maximal Schunck class. Then $|b(\underline{\underline{H}})| \leq 2$ by Lemma 2.31. If $|b(\underline{\underline{H}})| = 1$, then $\underline{\underline{H}}$ is maximal by Theorem 2.35 and so we may assume that $|b(\underline{\underline{H}})| = 2$. Let $b(\underline{\underline{H}}) = (B_1,B_2)$. We suppose $a(\underline{\underline{H}}) \neq b(\underline{\underline{H}})$ and choose G with minimal order in $a(\underline{\underline{H}})\backslash b(\underline{\underline{H}})$. Suppose G has just one of $\{B_1,B_2\}$ as a quotient, B_1 say. Then (G,B_2) is a Schunck boundary, $b(\underline{\underline{X}})$ say. Theorem 2.19 shows that $\underline{\underline{X}} \not\gg \underline{\underline{H}}$.

However $\underline{\underline{S}} >> h(B_2) >> \underline{\underline{X}} >> \underline{\underline{H}}$ is a proper chain of Schunck classes of length 3 and hence contradicts the 2-maximality of $\underline{\underline{H}}$.

Now suppose that G has B_1 and B_2 each appearing as quotients. Let $K \in \mathrm{Stab}(G)$, $p_1 \,\big|\, |F(B_1)|$ and $p_2 \,\big|\, |F(B_2)|$ with $p_1, p_2 \in \mathbb{P}$.

<u>Case (i)</u> If $p_1 \neq p_2$, then we assume without loss of generality that $p_1 \,\big|\, |F(G)|$. Let $Y \in \mathrm{Proj}_{h(B_1)}(K)$. Since $h(B_1) >> \underline{\underline{X}}$, all (G,Y)-pass groups lie in $b(\underline{\underline{H}})$ by choice of G and so all are isomorphic with B_1. Therefore $G \in a(h(B_1))$. This contradicts $a(h(B_1)) = (B_1)$ as proved in Lemma 2.34.

<u>Case (ii)</u> If $p_1 = p_2$ choose a maximal subgroup, X, of K containing an $h(B_2)$-projector of K. Then X has a quotient isomorphic with B_1 by Lemma 2.14. All (G,X)-pass groups lie in $a(\underline{\underline{H}})$, and hence in $b(\underline{\underline{H}})$ by the minimality of G, by Lemma 2.23. Let $\underline{\underline{F}}$ be the formation given by $(S \in \underline{\underline{S}} : \mathrm{Aut}_S(H/K) \not\cong B_1/F(B_1)$ for all p-chief factors H/K of $S)$. Then $X \notin \underline{\underline{F}}$ and so, by the usual arguments, some (G,X)-pass group has stabilizer not in $\underline{\underline{F}}$. Therefore $B_2 \notin \underline{\underline{F}}$ since $G \notin a(h(B_1)) = (B_1)$. It is easy to see that this forces $\gamma_q(B_2) \geq \gamma_q(B_1)$ for all primes q. We apply Lemma 1.29 to K with B_2 taking on the role of T. From this we are able to conclude that $\gamma_q(X) > \gamma_q(B_2)$ for some prime q distinct from p. But then some (G,X)-pass group must have Sylow q-subgroups with class greater than $\gamma_q(B_2) \geq \gamma_q(B_1)$. This impossible requirement completes the proof. $\square$

Theorems 2.35 and 2.36 suggest that the following may be true:

Conjecture: A Schunck class $\underline{\underline{H}}$ is n-maximal in $\underline{H}$ if and only if $|a(\underline{\underline{H}})| = |b(\underline{\underline{H}})| = n$.

The sufficiency of $|a(\underline{\underline{H}})| = |b(\underline{\underline{H}})| = n$ has already been dealt with but the necessity, even in the case $n = 3$, seems very difficult.

(2.37) Lemma

A Schunck class $\underline{\underline{X}}$ is uniquely determined by the maximal Schunck classes strongly containing it.

Proof Let $b(\underline{\underline{X}}) = (B_1, B_2, \ldots\ )$ and set $\underline{\underline{Y}}_i = h(B_i)$ for $i = 1,2,\ldots$. Certainly $\underline{\underline{Y}}_i >> \underline{\underline{X}}$ by Theorem 2.19, and $\underline{\underline{Y}}_i$ is maximal by Theorem 2.35. Lemma 2.22 shows that in fact $\underline{\underline{X}} = \bigwedge_{i=1}^{\infty} \underline{\underline{Y}}_i$ and the result is proved. □

E. Complements in $\underline{H}$.

In [6], Hawkes shows that $\underline{H}$ is a complemented lattice, that is, for every Schunck class $\underline{\underline{X}}$, there exists a Schunck class $\underline{\underline{Y}}$ with the properties $\underline{\underline{X}} \wedge \underline{\underline{Y}} = \underline{\underline{I}}$ and $\underline{\underline{X}} \vee \underline{\underline{Y}} = \underline{\underline{S}}$. Such complements are not in general uniquely determined as we shall see when we consider complements of $\underline{D}$ classes in Proposition 5.29.

For Schunck classes $\underline{\underline{X}}$, not of the form $\underline{\underline{Q}}^{\pi}$, for π a set of primes, and such that $b(\underline{\underline{X}}) \neq b_p(\underline{\underline{X}})$ for each prime p , we have the following recipe, due to Förster, for constructing a complement to $\underline{\underline{X}}$.

(2.α) : Since $\underline{\underline{X}} \neq \underline{\underline{Q}}^{\pi}$, there must be a non-cyclic $b(\underline{\underline{X}})$-group, B say. Let p be the prime dividing $|F(B)|$. Let $K \in \mathrm{Stab}(B)$. Hence $K \in \mathrm{Proj}_{\underline{\underline{X}}}(B)$. Since $b(\underline{\underline{X}}) \neq b_p(\underline{\underline{X}})$, there exists a $b_q(\underline{\underline{X}})$-group, T say, for some prime q distinct from p . Let $S \in \mathrm{Stab}(T)$. Set $\pi = \mathrm{char}(\underline{\underline{X}}) \cup (p,q)$. Let $J = \{1,2,\ldots..\}$ be an indexing set for π such that $p_1 = p$.

For each $n \in J$, set H_n to be the direct product of S with the direct product of n copies of B . Since B is non-abelian, the centralizers of pairs of minimal normal subgroups of $\underbrace{B \times \ldots \times B}_{n}$ differ. Hence the pairs of minimal normal subgroups of $\underbrace{B \times \ldots \times B}_{n}$ are not $(\underbrace{B \times \ldots \times B}_{n})$-isomorphic. It follows that the minimal normal subgroups of H_n are pairwise non-H_n isomorphic. We may regard $F(T)$ as an irreducible $Y_n = (\underbrace{K \times \ldots \times K}_{n}) \times S$-module with $\mathrm{Ker}(Y_n \text{ on } F(T)) =$

$(\underbrace{K \times \ldots \times K}_{n}) \times 1$. Then by Lemma 1.14 there is an irreducible $GF(q).H_n$-module, U_n say, faithful for H_n and such that there is a Y_n-submodule, $\bar{U}_n$ say, for which $[U_n/\bar{U}_n]Y_n/C_{Y_n}(U_n/\bar{U}_n) \cong T \in b(\underline{\underline{X}})$. Set $\bar{Y}_n = [\bar{U}_n]Y_n$. Similarly we may view $F(B)$ as an irreducible $\bar{Y}_n$-module with $\mathrm{Ker}(\bar{Y}_n$ on $F(B)) = [\bar{U}_n]\ ((1 \times \underbrace{K \times \ldots \times K}_{n-1}) \times S)$.

Since $[U_n]H_n$ has a unique minimal normal subgroup, we can apply Lemma 1.14 to obtain an irreducible $GF(p).[U_n]H_n$-module, V_n say, faithful for $[U_n]H_n$ and such that $V_n|_{\bar{Y}_n}$ has a $\bar{Y}_n$-submodule $\bar{V}_n$ satisfying $[V_n/\bar{V}_n]\bar{Y}_n/C_{\bar{Y}_n}(V_n/\bar{V}_n) \cong B \in b(\underline{\underline{X}})$. Set $L_n = [V_n]([U_n]H_n)$ and $\bar{L}_n = [\bar{V}_n]\bar{Y}_n$.

If $n \neq 1$, then $p_n \nmid |\bar{V}_n|$ and so Z_{p_n} is an irreducible $[\bar{V}_n]\bar{Y}_n$-module. Lemma 1.14 yields an irreducible $GF(p_n).L_n$-module, W_n say, faithful for L_n and such that W_n has an $\bar{L}_n$ submodule $\bar{W}_n$ with $\bar{L}_n$ acting trivially on $W_n/\bar{W}_n$. Set $G_n = [W_n]L_n$. By construction $[\bar{V}_n]\bar{Y}_n$ contains an $\underline{\underline{X}}$-projector for L_n , and so if $X_n \in \mathrm{Proj}_{\underline{\underline{X}}}(L_n)$, some (G_n, X_n)-pass group is isomorphic with Z_{p_n} .

Finally, if $n = 1$, we can take an irreducible $GF(q).L_1$-module, W_1 say, faithful for L_1 such that W_1 has an $\bar{L}_1$ module $\bar{W}_1$ with $[W_1/\bar{W}_1]\bar{L}_1/C_{\bar{L}_1}(W_1/\bar{W}_1) \cong T \in b(\underline{\underline{X}})$. Now there exists an irreducible

$GF(p).[W_1]L_1$-module, Z say, faithful for $[W_1]L_1$ and such that $Z_{[\bar{W}_1][\bar{V}_1]\bar{Y}_1}$ has the trivial $GF(p).[\bar{W}_1][\bar{V}_1]\bar{Y}_1$-module as a quotient. Thus Z_p appears as a $(G_1,[\bar{W}_1][\bar{V}_1]Y_1)$- pass group, where $G_1 = [Z][W_1]L_1$ and clearly $[\bar{W}_1][\bar{V}_1]\bar{Y}_1$ contains an $\underline{\underline{X}}$-projector for $[W_1]L_1$ and so Z_p appears as a (G_1,X_1)-pass group.

Set $b(\underline{\underline{X}}') = (G_i : i \in J)$. Then $a(\underline{\underline{X}}') = b(\underline{\underline{X}}')$ and $\underline{\underline{X}}'$ complements $\underline{\underline{X}}$ in $\underline{H}$. (For a proof see Förster [2], Satz 8.9.)

§3. The Classes $b'(\underline{\underline{X}})$ and $\underline{\underline{X}}^{\prec}$.

Here we give two recipes for constructing Schunck boundaries (and the corresponding Schunck classes) from any given class of primitive groups.

(3.1) Definition

Let $\underline{\underline{X}} \subseteq \underline{\underline{P}}$ and let $\prec$ be a partial ordering on $\underline{\underline{X}}$ given by $X \prec Y$ if and only if $X \in Q(Y)$. We set $\underline{\underline{X}}^{\prec}$ to be those elements of $\underline{\underline{X}}$ minimal with respect to $\prec$.

It is clear that for any $\underline{\underline{X}} \subseteq \underline{\underline{P}}$, the class $\underline{\underline{X}}^{\prec}$ is always a Schunck boundary. If $\underline{\underline{H}}$ is a Schunck class, then it is easy to see that $(a(\underline{\underline{H}}))^{\prec} = b(\underline{\underline{H}})$.

We now use this construction to express any Schunck class $\underline{\underline{X}}$ as the meet of Schunck classes $\underline{\underline{X}}_p$ with $b(\underline{\underline{X}}_p) = b_p(\underline{\underline{X}}_p)$ for each prime p.

(3.2) Lemma

Let $\underline{\underline{X}}$ be a Schunck class. Then for each prime p there exists a Schunck class $\underline{\underline{X}}_p$ with $b(\underline{\underline{X}}_p) = (a_p(\underline{\underline{X}}))^{\prec}$ and whose avoidance class coincides with $a_p(\underline{\underline{X}})$.

Proof Let $\underline{\underline{X}}_p$ be the Schunck class with boundary $(a_p(\underline{\underline{X}}))^{\prec}$. Then $b(\underline{\underline{X}}_p) \subseteq a_p(\underline{\underline{X}})$ and so $\underline{\underline{X}}_p >> \underline{\underline{X}}$. Furthermore $a(\underline{\underline{X}}_p) \subseteq a_p(\underline{\underline{X}})$.

Suppose $a_p(\underline{\underline{X}}) \neq a(\underline{\underline{X}}_p)$ and choose G with minimal order from $a_p(\underline{\underline{X}}) \setminus a(\underline{\underline{X}}_p)$. Then, by definition of $b(\underline{\underline{X}}_p)$, it follows that G has a quotient in $b(\underline{\underline{X}}_p)$. Call the quotient B , then $B \in a_p(\underline{\underline{X}})$. Therefore $h(B) >> \underline{\underline{X}}$ and so, if $L \in \mathrm{Proj}_{h(B)}(S)$ for $S \in \mathrm{Stab}(G)$, then all (G,L)-pass groups lie in $a_p(\underline{\underline{X}})$ by Lemma 2.23. Therefore all (G,L)-pass groups lie in $a(\underline{\underline{X}}_p)$ by the minimality of G . It follows from Lemma 2.25 that $G \in a(\underline{\underline{X}}_p)$ and hence $a_p(\underline{\underline{X}}) = a(\underline{\underline{X}}_p)$. □

(3.3) Corollary

If $\underline{\underline{X}}$ is a Schunck class, then $\underline{\underline{X}} = \bigwedge_{p \in \mathbb{P}} \underline{\underline{X}}_p$ where $\underline{\underline{X}}_p = h((a_p(\underline{\underline{X}}))^{\neg})$.

Proof Since $\underline{\underline{X}}_p >> \underline{\underline{X}}$ for all primes p , we have $\bigwedge_{p \in \mathbb{P}} \underline{\underline{X}}_p >> \underline{\underline{X}}$. Suppose $\wedge \underline{\underline{X}}_p \neq \underline{\underline{X}}$, then there is a group G in $\wedge \underline{\underline{X}}_p \setminus \underline{\underline{X}}$. Therefore G has a quotient in $b(\underline{\underline{X}})$ and hence in $b_p(\underline{\underline{X}})$ for some prime p . Then $G \notin \underline{\underline{X}}_p$ since $b_p(\underline{\underline{X}}) \subseteq b(\underline{\underline{X}}_p)$. Hence, in particular, $G \notin \wedge \underline{\underline{X}}_p$. This contradiction completes the proof. □

(3.4) Definition

Let $\underline{\underline{X}} \subseteq \underline{\underline{P}}$. Let $\underline{\underline{X}}_0 = \underline{\underline{X}}$ and define recursively $\underline{\underline{X}}_i$ by:

$$\underline{\underline{X}}_i = ((G,K)\text{-pass groups: } G \in \underline{\underline{X}}_{i-1} \text{ , } J \in \mathrm{Proj}_{h(\underline{\underline{X}}_{i-1})}(S)$$

$$\text{for } S \in \mathrm{Stab}(G) \text{ and } J \leq K \leq S \text{ .)}$$

Let $\underline{\underline{X}}_\infty = \bigcup_{i=0}^{\infty} \underline{\underline{X}}_i$ and $b'(\underline{\underline{X}}) = (\underline{\underline{X}}_\infty)^{\prec}$. Then $b'(\underline{\underline{X}})$ is a Schunck boundary.

(3.5) Proposition

If $\underline{\underline{X}}$ is any class of primitive groups, then $\underline{\underline{X}} \subseteq \underline{\underline{X}}_\infty \subseteq a(\underline{\underline{Y}})$ where $\underline{\underline{Y}}$ is the Schunck class having boundary $b'(\underline{\underline{X}})$.

Proof Clearly $\underline{\underline{X}} \subseteq \underline{\underline{X}}_\infty$. We now show $\underline{\underline{X}}_\infty \subseteq a(\underline{\underline{Y}})$. If G has minimal order in $\underline{\underline{X}}_\infty$, then certainly $G \in (\underline{\underline{X}}_\infty)^{\prec} = b(\underline{\underline{Y}}) \subseteq a(\underline{\underline{Y}})$. Suppose now that for an $\underline{\underline{X}}_\infty$-group H , all $\underline{\underline{X}}_\infty$-groups with order less than $|H|$ lie in $a(\underline{\underline{Y}})$. Since $H \in \underline{\underline{X}}_\infty$, it follows that H has a quotient in $(\underline{\underline{X}}_\infty)^{\prec} = b(\underline{\underline{Y}})$. Call this quotient B . Then $B \in \underline{\underline{X}}_\infty$ and so $B \in \underline{\underline{X}}_r$ for all r greater than some $m \in \mathbb{N}$. Similarly $H \in \underline{\underline{X}}_r$ for all r greater than some $n \in \mathbb{N}$. Let $t = \max\{m,n\}$. Then $(B,H) \subseteq \underline{\underline{X}}_t$. Let $T \in \mathrm{Proj}_{h(B)}(S)$ where $S \in \mathrm{Stab}(H)$. If $T = S$, then $B = H$ and so $H \in b(\underline{\underline{Y}}) \subseteq a(\underline{\underline{Y}})$. We now assume $|T| < |S|$. Since $B \in b(\underline{\underline{Y}})$ and $B \in \underline{\underline{X}}_t$, we have $B \in (\underline{\underline{X}}_t)^{\prec}$. Therefore $h(B) >> h(\underline{\underline{X}}_t)$ and so all (H,T)-pass groups lie in $\underline{\underline{X}}_{t+1}$ by definition, and hence all lie in $\underline{\underline{X}}_\infty$ and then in $a(\underline{\underline{Y}})$ by hypothesis since $|T| < |S|$. Now, by Lemma 2.25, we have $H \in a(\underline{\underline{Y}})$. □

(3.6) Proposition

Let $\underline{\underline{X}}$ be any class of primitive groups and let $\underline{\underline{Y}}$ be the Schunck class with boundary $b'(\underline{\underline{X}})$. Then $\underline{\underline{Y}}$ is the unique maximal Schunck class with $\underline{\underline{X}}$ contained in its avoidance class.

Proof Let $\underline{\underline{Z}}$ be a Schunck class with $\underline{\underline{X}} \subseteq a(\underline{\underline{Z}})$. We aim to prove that $\underline{\underline{Y}} >> \underline{\underline{Z}}$. We begin by showing $\underline{\underline{X}}_\infty \subseteq a(\underline{\underline{Z}})$. Assume that $\underline{\underline{X}}_n \subseteq a(\underline{\underline{Z}})$ for $n \in \mathbb{N}$. Let $H \in \underline{\underline{X}}_{n+1}$. Then H is a (G,K)-pass group for some $G \in \underline{\underline{X}}_n$, $S \in \mathrm{Stab}(G)$, $J \in \mathrm{Proj}_{h(\underline{\underline{X}}_n)}(S)$ and $J \leq K \leq S$. Now $G \in a(\underline{\underline{Z}})$ since $G \in \underline{\underline{X}}_n$. Also, since $\underline{\underline{X}}_n \subseteq a(\underline{\underline{Z}})$, we see that $h(\underline{\underline{X}}_n) >> \underline{\underline{Z}}$ and so, by Lemma 2.23, all (G,K)-pass groups lie in $a(\underline{\underline{Z}})$. Therefore $H \in a(\underline{\underline{Z}})$ by Lemma 2.25. Hence $\underline{\underline{X}}_{n+1} \subseteq a(\underline{\underline{Z}})$. It follows that $\underline{\underline{X}}_\infty \subseteq a(\underline{\underline{Z}})$, hence $b(\underline{\underline{Y}}) = (\underline{\underline{X}}_\infty)^{\prec} \subseteq a(\underline{\underline{Z}})$, and so finally $\underline{\underline{Y}} >> \underline{\underline{Z}}$. $\square$

(3.7) Examples (i) If $\underline{\underline{X}}$ is a Schunck class, then $b'(b(\underline{\underline{X}})) = b(\underline{\underline{X}})$ since $(b(\underline{\underline{X}}))_0 = (b(\underline{\underline{X}}))_1 = \ldots\ldots$. Furthermore, it is easy to see by Lemma 2.23 that $(a(\underline{\underline{X}}))_0 = (a(\underline{\underline{X}}))_1 = \ldots\ldots$ and so $b'(a(\underline{\underline{X}})) = (a(\underline{\underline{X}}))^{\prec} = b(\underline{\underline{X}})$.

(ii) Let H be a polyprimitive group of type $(2,3,5)$. Let $\underline{\underline{X}} = (H, Z_5, S_3)$. Then $\underline{\underline{X}}_1 = (H, Z_5, S_3, E(2/5)) = \underline{\underline{X}}_2 = \ldots\ldots$. Therefore $b'(\underline{\underline{X}}) = (Z_5, S_3, E(2/5))$.

(3.8) Lemma

Let $\underline{\underline{X}}$ be a Schunck class and let $\underline{\underline{Y}}$ be any class of primitive groups. Then $b'(b(\underline{\underline{X}}) \cup \underline{\underline{Y}}) = b'(a(\underline{\underline{X}}) \cup \underline{\underline{Y}})$.

Proof Let $\underline{\underline{Z}}$ and $\underline{\underline{W}}$ be Schunck classes satisfying $b(\underline{\underline{Z}}) = b'(b(\underline{\underline{X}}) \cup \underline{\underline{Y}})$

and $b(\underline{\underline{W}}) = b'(a(\underline{\underline{X}}) \cup \underline{\underline{Y}})$. Then $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{Z}})$ by Proposition 3.5. Therefore $a(\underline{\underline{X}}) \subseteq a(\underline{\underline{Z}})$ by Theorem 2.19, and hence $a(\underline{\underline{X}}) \cup \underline{\underline{Y}} \subseteq a(\underline{\underline{Z}})$. It follows by Proposition 3.6 that $\underline{\underline{W}} >> \underline{\underline{Z}}$.

Now $b(\underline{\underline{X}}) \cup \underline{\underline{Y}} \subseteq a(\underline{\underline{X}}) \cup \underline{\underline{Y}} \subseteq a(\underline{\underline{W}})$ by Proposition 3.5. Then, by Proposition 3.6, we have $\underline{\underline{Z}} >> \underline{\underline{W}}$, and hence $\underline{\underline{Z}} = \underline{\underline{W}}$. □

To conclude this section we show how these constructions relate to the meet and join operations in the lattice $\underline{H}$.

(3.9) <u>Lemma</u>

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are Schunck classes, then

(i) $$b(\underline{\underline{X}} \vee \underline{\underline{Y}}) = (a(\underline{\underline{X}}) \cap a(\underline{\underline{Y}}))$$

(ii) $$b(\underline{\underline{X}} \wedge \underline{\underline{Y}}) = b'(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})) \ .$$

<u>Proof</u> (i) See Lemma 2.22.

(ii) This follows from the definition of the meet operation and Proposition 3.6. □

We now characterize those Schunck classes which are meet-irreducible in $\underline{H}$.

(3.10) <u>Proposition</u>

Let $\underline{\underline{H}}$ be a Schunck class. Then $\underline{\underline{H}}$ is meet-irreducible in $\underline{H}$

if and only if $\underline{\underline{H}}$ is maximal in $\underline{H}$ or $\underline{\underline{H}} = \underline{\underline{S}}$.

Proof Let T be any $b(\underline{\underline{H}})$-group. Define a Schunck class $\underline{\underline{X}}$ by $b(\underline{\underline{X}}) = b(\underline{\underline{H}})\backslash(T)$. By Lemma 3.9, $b(h(T) \wedge \underline{\underline{X}}) = b'((T) \cup b(\underline{\underline{X}})) = b'(b(\underline{\underline{H}})) = b(\underline{\underline{H}})$. Therefore $\underline{\underline{H}} = h(T) \wedge \underline{\underline{X}}$. Now $h(T) \not\gg \underline{\underline{X}}$ since $T \notin a(\underline{\underline{X}})$, and $\underline{\underline{X}} \not\gg h(T)$ unless $b(\underline{\underline{X}}) = \phi$. Therefore $\underline{\underline{H}}$ is meet-reducible if $|b(\underline{\underline{H}})| > 1$. Hence the meet-irreducible Schunck classes are those with boundaries of size 0 or 1 , that is $\underline{\underline{S}}$ and the maximal Schunck classes. □

A characterization of the join-irreducible Schunck classes appears much more difficult to obtain. Clearly any atoms of $\underline{H}$ are join irreducible but we now see that there are join-irreducible Schunck classes which are not atoms.

(3.11) Example

Let $\underline{\underline{X}}$ be the Schunck class with boundary $(a(\underline{\underline{S}}_{p'})\backslash(Z_p))^{<}$ for some prime p . Then $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{S}}_{p'})$ and hence $\underline{\underline{X}} \gg \underline{\underline{S}}_{p'}$ by Theorem 2.19. Now $a(\underline{\underline{S}}_{p'})$ consists of all primitive groups G whose Hall p'-subgroups intersect $Soc(G)$ trivially. Therefore $a(\underline{\underline{S}}_{p'}) = (G \in \underline{\underline{P}} : F(G)$ is a p-group). Suppose $\underline{\underline{Y}} \underset{\neq}{\ll} \underline{\underline{X}}$ for some Schunck class $\underline{\underline{Y}}$. Let G be a group of minimal order in $\underline{\underline{X}}\backslash\underline{\underline{Y}}$. Then $G \in \underline{\underline{X}} \cap b(\underline{\underline{Y}})$. Either G is isomorphic with Z_p or $F(G)$ is a p'-group for if G is non-cyclic and p divides $|F(G)|$ then $G \in a(\underline{\underline{S}}_{p'})\backslash(Z_p)$ and so G has a quotient in $b(\underline{\underline{X}})$ and hence $G \notin \underline{\underline{X}}$. We consider the latter case.

Let $S \in \mathrm{Stab}(G)$. By Lemma 1.14 there is an irreducible $GF(p).G$ -module V faithful for G such that $1_S \in Q(V_S)$. Now $[V]G$ has socle a p-group and so $[V]G \in a(\underline{\underline{S}}_{p'})$ hence $[V]G$ has a quotient in $b(\underline{\underline{X}})$. Since $G \in \underline{\underline{X}}$, we have $[V]G \in b(\underline{\underline{X}})$. By Proposition 3.6 we have $b'(b(\underline{\underline{X}}) \cup (G)) \subseteq a(\underline{\underline{Y}})$. Now, by the choice of V, we have $Z_p \in b(\underline{\underline{Y}})$. If $G \cong Z_p$, then of course $Z_p \in b(\underline{\underline{Y}})$.

We have shown that for all Schunck classes $\underline{\underline{Y}}$ properly strongly contained in $\underline{\underline{X}}$, it is necessary that $Z_p \in b(\underline{\underline{Y}})$. Suppose $\underline{\underline{X}} = \bigvee_{i \in I} \underline{\underline{Y}}_i$ for a family $\{\underline{\underline{Y}}_i\}_{i \in I}$ of Schunck classes properly strongly contained in $\underline{\underline{X}}$, then $Z_p \in b(\underline{\underline{Y}}_i)$ for each $i \in I$ and so $Z_p \in \bigcap_{i \in I} a(\underline{\underline{Y}}_i) = a(\underline{\underline{X}})$. Clearly, by the definition of $\underline{\underline{X}}$, this cannot hold and so $\underline{\underline{X}}$ must be join irreducible but $\underline{\underline{X}} >> \underline{\underline{S}}_{p'}$, and so $\underline{\underline{X}}$ is not an atom. $\square$

This example also shows that a Schunck class is not uniquely determined by the set of atoms it strongly contains since $h((a(\underline{\underline{S}}_{p'}) \setminus (Z_p)))$ and $\underline{\underline{S}}_{p'}$ contain strongly the same atoms. Therefore the dual to Lemma 2.37 does not hold.

§4. Schunck classes $\underline{\underline{H}}$ with only finite chains from $\underline{\underline{H}}$ to $\underline{\underline{S}}$

(4.1) Lemma

Let $\underline{\underline{H}}$ be a Schunck class with $b(\underline{\underline{H}}) = b_p(\underline{\underline{H}})$ for some prime p. Let A be an $(a(\underline{\underline{H}})\backslash b(\underline{\underline{H}}))$-group and $T \in Q(A) \cap b(\underline{\underline{H}})$. If $H \in \mathrm{Proj}_{\underline{\underline{H}}}(A)$, there exists a prime $q \neq p$ for which $\gamma_q(H) > \gamma_q(T)$. Therefore there is a $b(\underline{\underline{H}})$-group B with $\gamma_q(B) > \gamma_q(T)$.

If, in addition, $\ell = \ell(T)$ is maximal among the nilpotent lengths of $b(\underline{\underline{H}})$-groups, then $L_{\ell-2}(T)/L_{\ell-1}(T)$ is an r-group for some prime r and we can take $q = r$ above.

Proof Let $G \in \mathrm{Stab}(A)$. We have by Lemma 1.29 that $\gamma_q(A) = \gamma_q(G) > \gamma_q(T)$ for some prime $q \neq p$. Since $b(\underline{\underline{H}}) = b_p(\underline{\underline{H}})$, it follows that $\gamma_q(H) = \gamma_q(A)$. Now all (A,H)-pass groups lie in $b(\underline{\underline{H}})$ and by Corollary 2.27 there is such a pass group, B say, with $\gamma_q(B) = \gamma_q(H) > \gamma_q(T)$.

We now assume that the nilpotent length of T is maximal among all $b(\underline{\underline{H}})$-groups. Let $N \trianglelefteq G$ be such that G/N is isomorphic with T. As in the proof of Lemma 1.29 there exists a prime q for which $G/C_G(O_q(G)) \notin \underline{\underline{E}}$ where $\underline{\underline{E}}$ is given by (1.α) (page 18). Set $U = C_G(O_q(G))$. Thus $U \leq N$. Take $M \trianglelefteq G$ such that $M/N \cong F(T)$. Hence HN/N complements M/N in G/N. Consider $O_q(G)$ as an HU-group. If V is any HU-composition factor of $O_q(G)$, then since $\bar{H} := HU/\mathrm{Ker}(HU \text{ on } V) \in Q(H)$ and $H \in R_0\, c(\underline{\underline{H}})$ we have $\ell(\bar{H}) \leq \ell(H)-1 = \ell-2$.

Therefore $L_{\ell-2}(H)U/U$ centralizes V and hence it stabilizes a series for $O_q(G)$. It follows by Lemma 1.6 that $L_{\ell-2}(H)U/U$ is a q-group. Recall that $G = HM$ and G/M is isomorphic with $T/F(T)$. Hence $L_{\ell-2}(T)/L_{\ell-1}(T) \cong L_{\ell-2}(T/F(T)) = L_{\ell-2}(G/M) \cong L_{\ell-2}(HM/M) \cong L_{\ell-2}(H).M/M \cong L_{\ell-2}(H)/M \cap L_{\ell-2}(H) \in Q(L_{\ell-2}(H)/L_{\ell-2}(H) \cap U) = Q(L_{\ell-2}(H)U/U)$ and $L_{\ell-2}(T)/L_{\ell-1}(T)$ is a q-group. □

(4.2) Proposition

If $\underline{\underline{H}}$ is a Schunck class with an infinite avoidance class, then $a(\underline{\underline{H}})$ contains an infinite Schunck boundary.

Proof We first prove the result for Schunck classes $\underline{\underline{H}}$ satisfying $b(\underline{\underline{H}}) = b_p(\underline{\underline{H}})$ for some prime p.

Here we use induction on $|b(\underline{\underline{H}})|$ which we may clearly assume to be finite. If $|b(\underline{\underline{H}})| = 1$, then $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$ by Lemma 2.34 and so this case cannot arise. Now let $|b(\underline{\underline{H}})| = n > 1$ and assume the result is true for all Schunck classes having at most $n-1$ elements in their boundaries. For each prime q, let $c(q) = \max\{\gamma_q(G) : G \in c(\underline{\underline{H}})\}$ where we take $\gamma_q(G) = 0$ if $q \nmid |G|$. We define, for each prime q, the class $\underline{\underline{A}}_{c(q)}$ given by

$$\underline{\underline{A}}_{c(q)} = \begin{cases} \phi, & \text{if } c(q) = 0 \text{ or } q = p \\ (G \in a(\underline{\underline{H}}) : \gamma_q(G) = c(q)), & \text{otherwise.} \end{cases}$$

Since $c(\underline{\underline{H}})$ is by assumption finite, $c(q) \neq 0$ (and hence $\underline{\underline{A}}_{c(q)} \neq \phi$) for only finitely many primes, $q_1, \ldots, q_n$ say. Let $\underline{\underline{L}}_0 = a(\underline{\underline{H}})$ and for $i = 1,2,\ldots,n$ let $\underline{\underline{L}}_i = \underline{\underline{L}}_{i-1} \cap \underline{\underline{A}}_{c(q_i)}$.

Case (a) :

Suppose $\underline{\underline{L}}_n$ is infinite. We show that $\underline{\underline{L}}_n$ is a Schunck boundary. If not, there exist groups $G,H \in \underline{\underline{L}}_n$ with $G \in Q(H)$. In this case each of G and H have Sylow q-subgroups of maximal class among $a(\underline{\underline{H}})$-groups for all primes q distinct from p since $\max(\gamma_q(A) : A \in a(\underline{\underline{H}})) = c(q)$ by Corollary 2.27. However, Lemma 1.29 gives $\gamma_q(H) > \gamma_q(G)$ for some prime q distinct from p . From this contradiction it is clear that $\underline{\underline{L}}_n$ is an infinite Schunck boundary contained in $a(\underline{\underline{H}})$.

Case (b) :

Now we assume that $\underline{\underline{L}}_n$ is finite. Let i be the smallest suffix such that $\underline{\underline{L}}_i$ is finite. Then $i \geq 1$. Let $\underline{\underline{R}} = (B \in b(\underline{\underline{H}}) : \gamma_{q_i}(B) = c(q_i))$ and so $R \neq \phi$ by definition of $c(q_i)$. Set $b(\underline{\underline{X}}) = b(\underline{\underline{H}}) \backslash \underline{\underline{R}}$ and let G be a group in $\underline{\underline{L}}_{i-1} \backslash \underline{\underline{L}}_i$. Then no $\underline{\underline{R}}$-group appears as a cast-off group for G , since, if some $\underline{\underline{R}}$-group was a cast-off group, then we must have $\gamma_{q_i}(G) \geq c(q_i)$ but then by Corollary 2.7, $\gamma_{q_i}(G) = c(q_i)$ and we have $G \in \underline{\underline{L}}_i$. Therefore all the cast-off groups for G lie in $b(\underline{\underline{X}})$ and so $G \in a(\underline{\underline{X}})$ by Lemma 2.25. Then $b(\underline{\underline{X}}) \neq \phi$ and $\underline{\underline{L}}_{i-1} \setminus \underline{\underline{L}}_i \subsetneq a(\underline{\underline{X}})$. Now $b(\underline{\underline{X}}) \subset b(\underline{\underline{H}})$ implies $a(\underline{\underline{X}}) \subset a(\underline{\underline{H}})$. Since $\underline{\underline{L}}_{i-1} \setminus \underline{\underline{L}}_i$ is infinite, it follows that $a(\underline{\underline{X}})$ is infinite. Since $|b(\underline{\underline{X}})| < |b(\underline{\underline{H}})|$, it now follows by induction that $a(\underline{\underline{X}})$, and hence $a(\underline{\underline{H}})$, contains an infinite Schunck boundary.

Now we prove the result for any Schunck class $\underline{\underline{H}}$. Again we may assume that $|b(\underline{\underline{H}})|$ is finite. Therefore $b_p(\underline{\underline{H}}) \neq \phi$ for only finitely many primes p. Consider a prime p for which $a_p(\underline{\underline{H}})$ is infinite. It is clear that such a prime exists, for $a_p(\underline{\underline{H}}) = \phi$ if $b_p(\underline{\underline{H}}) = \phi$. Let $b(\underline{\underline{X}}) = (a_p(\underline{\underline{H}}))^{\prec}$. By Lemma 3.2, $a_p(\underline{\underline{H}})$ is the avoidance class for $\underline{\underline{X}}$, and since $b(\underline{\underline{X}}) = b_p(\underline{\underline{X}})$, we conclude from the special case above that $a_p(\underline{\underline{H}})$, and hence $a(\underline{\underline{H}})$, contains an infinite Schunck boundary. □

(4.3) <u>Corollary</u>

A Schunck class $\underline{\underline{H}}$ has no infinite proper chains from $\underline{\underline{H}}$ to $\underline{\underline{S}}$ if and only if $a(\underline{\underline{H}})$ is finite.

<u>Proof</u> If $a(\underline{\underline{H}})$ is infinite, then it contains an infinite Schunck boundary by Proposition 4.3, $b(\underline{\underline{X}})$ say. Let B_1 be any $b(\underline{\underline{X}})$-group and set $b(\underline{\underline{X}}_1) = b(\underline{\underline{X}}) \setminus (B_1)$. Define $\underline{\underline{X}}_2, \ldots\ldots$ by letting B_i be any $b(\underline{\underline{X}}_{i-1})$-group and setting $b(\underline{\underline{X}}_i) = b(\underline{\underline{X}}_{i-1}) \setminus (B_i)$. This recipe gives an infinite proper chain $\underline{\underline{H}} << \underline{\underline{X}} << \underline{\underline{X}}_1 << \ldots . << \underline{\underline{S}}$.

Conversely, if $a(\underline{\underline{H}})$ is finite, then all chains have length at most $|a(\underline{\underline{H}})|$. For, if $\underline{\underline{X}} \underset{\neq}{>>} \underline{\underline{H}}$, then $a(\underline{\underline{X}}) \subsetneq a(\underline{\underline{H}})$. □

Finally in this section we show that those Schunck classes having no infinite proper chains to $\underline{\underline{S}}$ do not form a sublattice of $\underline{H}$.

(4.5) Example

Certainly $\underline{\underline{Q}}^p$, $\underline{\underline{Q}}^q$ have no infinite proper chains to $\underline{\underline{S}}$ where p and q are distinct primes. Now $b(\underline{\underline{Q}}^p \wedge \underline{\underline{Q}}^q) = (Z_p, Z_q)$ by Lemma 2.22. However, for each $n \in \mathbb{N}$, if H_n is a cyclic group of order q^n and V_n is a faithful, irreducible $GF(p).H_n$-module, then $[V_n].H_n \in a(\underline{\underline{Q}}^p \wedge \underline{\underline{Q}}^q)$. Then setting $b(\underline{\underline{X}}_m) = ([V_0]H_0, \ldots, [V_m]H_m)$ for each $m \in \mathbb{N}$ yields a properly descending chain, $\underline{\underline{S}} >> \underline{\underline{X}}_0 >> \underline{\underline{X}}_1 >> \ldots\ldots >> \underline{\underline{Q}}^p \wedge \underline{\underline{Q}}^q$, of infinite length. □

§5. $\underline{D}$-boundaries.

We first consider some necessary and sufficient conditions for a Schunck boundary to be a $\underline{D}$-boundary.

(5.1) Definitions

Let G be a primitive group with stabilizer K.

If $\underline{\underline{X}}$ is a class of groups, then we may refer to H as a $(G,\underline{\underline{X}})$-*pass group* if it is a (G,J)-pass group for some $\underline{\underline{X}}$-subgroup J of K.

A G-pass group H will be described as *proper* if $G \neq H$.

(5.2) Theorem (Förster)

A Schunck class $\underline{\underline{D}}$ is a $\underline{D}$-class if and only if for each group G in $b(\underline{\underline{D}})$ all $(G,\underline{\underline{D}})$-pass groups lie in $b(\underline{\underline{D}})$.

Proof Förster [2], 9.4. □

(5.3) Theorem

A Schunck class $\underline{\underline{D}}$ is a $\underline{D}$-class if and only if for each group G in $a(\underline{\underline{D}})$ all G-pass groups lie in $a(\underline{\underline{D}})$.

Proof Let $G \in a(\underline{\underline{D}})$ and let W be a G-pass group. Then W has the form $[R/S]\ J/C_J(R/S)$ where J is a subgroup of $K \in \mathrm{Stab}(G)$ and R/S is a J-composition factor of $F(G)$. We set $H = RJ$ and $N = SC_J(R/S)$. Then $N \trianglelefteq H$ and H/N is isomorphic with W. Let $D \in \mathrm{Proj}_{\underline{\underline{D}}}(H)$. Since $\underline{\underline{D}}$ is a $\underline{D}$-class, there is a $\underline{\underline{D}}$-projector of G containing D. However $G \in a(\underline{\underline{D}})$ and so all $\underline{\underline{D}}$-projectors avoid $F(G)$. Therefore $D \cap F(G) = 1$ and hence $D \cap R = 1$. Let $X \in \mathrm{Proj}_{\underline{\underline{D}}}(J)$. Then $X \in \mathrm{Proj}_{\underline{\underline{D}}}(H)$ since $D \cong D/D \cap R = DR/R \in \mathrm{Proj}_{\underline{\underline{D}}}(RJ/R)$ and $RJ/R \cong J$. It follows that $XN/N \in \mathrm{Proj}_{\underline{\underline{D}}}(H/N)$ and since $XN/N \leq JN/N \in \mathrm{Stab}(H/N)$ we have $H/N \in a(\underline{\underline{D}})$ as required.

The converse follows immediately from Theorem 5.2. □

Armed with these two results, we can now describe two $\underline{D}$-boundaries, associated with any class $\underline{\underline{X}}$ of primitive groups, which are similar to $\underline{\underline{X}}^{\prec}$ and $b'(\underline{\underline{X}})$ described earlier. We shall see that in the special case of $\underline{\underline{X}}$ being a Schunck boundary, $b(\underline{\underline{Y}})$ say, then these constructions give rise to $\underline{D}$-boundaries corresponding to the unique maximal $\underline{D}$-class strongly contained in $\underline{\underline{Y}}$ and the unique minimal $\underline{D}$-class strongly containing $\underline{\underline{Y}}$.

(5.4) Definition

Let $\underline{\underline{X}}$ be a class of primitive groups. Define $\underline{p(\underline{\underline{X}})}$ to be the class consisting of all G-pass groups for all $\underline{\underline{X}}$-groups G.

Let $b^*(\underline{\underline{X}}) = (p(\underline{\underline{X}}))^{\prec}$.

(5.5) Lemma

If $\underline{\underline{X}}$ is a class of primitive groups, then $b^*(\underline{\underline{X}})$ is a $\underline{D}$-boundary.

Proof It is clear that $b^*(\underline{\underline{X}})$ is a Schunck boundary, $b(\underline{\underline{Y}})$ say. Let G be a $b(\underline{\underline{Y}})$-group with $K \in \mathrm{Stab}(G)$ and J a $\underline{\underline{Y}}$-subgroup of K. Let W be a (G,J)-pass group. Then $W \in p(\underline{\underline{X}})$. If $W \notin b(\underline{\underline{Y}})$, then W must have a proper quotient in $b(\underline{\underline{Y}})$. It follows that J must have a quotient in $b(\underline{\underline{Y}})$ but this contradicts $J \in \underline{\underline{Y}}$. Therefore $W \in b(\underline{\underline{Y}})$ and so $b(\underline{\underline{Y}})$ is a $\underline{D}$-boundary by Theorem 5.2. □

(5.6) Definition

For a Schunck class $\underline{\underline{X}}$ define the $\underline{D}$-*interior* by $\underline{\underline{X}}^0 = h(b^*(b(\underline{\underline{X}})))$.

(5.7) Lemma

Let $\underline{\underline{X}}$ be a Schunck class. Then $\underline{\underline{X}}^0$ is a $\underline{D}$-class strongly contained in $\underline{\underline{X}}$.

Proof That $\underline{\underline{X}}^0$ is a $\underline{D}$-class follows from Lemma 5.5. Let G be a $b(\underline{\underline{X}})$-group and $K \in \mathrm{Stab}(G)$. Let J be an $\underline{\underline{X}}^0$-projector of K and let L be a (G,J)-pass group. Then $L \in p(b(\underline{\underline{X}}))$ but L has no proper quotient in $b^*(b(\underline{\underline{X}}))$ since $L/F(L) \in Q(J) \subseteq \underline{\underline{X}}^0 = h(b^*(b(\underline{\underline{X}})))$. Therefore $L \in b^*(b(\underline{\underline{X}})) = b(\underline{\underline{X}}^0)$. Therefore $G \in a(\underline{\underline{X}}^0)$ by Lemma 2.24.

Now $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{X}}^0)$ yields $\underline{\underline{X}} \gg \underline{\underline{X}}^0$ by Theorem 2.19. □

(5.8) Lemma

A Schunck class $\underline{\underline{D}}$ is a $\underline{D}$-class if and only if $\underline{\underline{D}} = \underline{\underline{D}}^0$.

Proof If $\underline{\underline{D}} = \underline{\underline{D}}^0$, we have, by Lemma 5.5, that $\underline{\underline{D}}$ is a $\underline{D}$-class.

Conversely, let $\underline{\underline{D}}$ be a $\underline{D}$-class. We have, by Lemma 5.7, that $\underline{\underline{D}}^0 \ll \underline{\underline{D}}$ and hence $\underline{\underline{D}}^0 \subseteq \underline{\underline{D}}$. Suppose that $\underline{\underline{D}}^0 \neq \underline{\underline{D}}$ and choose G with minimal order in $\underline{\underline{D}} \backslash \underline{\underline{D}}^0$. Then G has a quotient in $b(\underline{\underline{D}}^0)$. Now $G \in \underline{\underline{D}}$ implies $Q(G) \subseteq \underline{\underline{D}}$, hence, by the minimality of G , we must have $G \in b(\underline{\underline{D}}^0) = b^*(b(\underline{\underline{D}})) \subseteq p(b(\underline{\underline{D}})) \subseteq a(\underline{\underline{D}})$ by Theorem 5.3. Therefore $G \in a(\underline{\underline{D}})$ and hence $G \notin \underline{\underline{D}}$. This contradiction yields the required result. □

(5.9) Proposition

Let $\underline{\underline{X}}$ be a Schunck class. Then $\underline{\underline{X}}^0$ is the unique maximal $\underline{D}$-class strongly contained in $\underline{\underline{X}}$.

Proof By Lemma 5.7, we have $\underline{\underline{X}}^0 \ll \underline{\underline{X}}$. We take a $\underline{D}$-class $\underline{\underline{Y}}$ strongly contained in $\underline{\underline{X}}$ and show that $\underline{\underline{X}}^0 \gg \underline{\underline{Y}}$. If $\underline{\underline{X}} \gg \underline{\underline{Y}}$, it follows that $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{Y}})$ by Theorem 2.19. Hence $p(b(\underline{\underline{X}})) \subseteq p(a(\underline{\underline{Y}}))$ and since $\underline{\underline{Y}} \in \underline{D}$, we have $p(a(\underline{\underline{Y}})) = a(\underline{\underline{Y}})$ by Theorem 5.3. Therefore $b(\underline{\underline{X}}^0) = b^*(b(\underline{\underline{X}})) \subseteq p(b(\underline{\underline{X}})) \subseteq a(\underline{\underline{Y}})$ and hence $\underline{\underline{X}}^0 \gg \underline{\underline{Y}}$ by Theorem 2.19. □

There is clearly considerable similarity between $b'(\underline{\underline{X}})$ and $b^*(\underline{\underline{X}})$. The following result should be compared with Proposition 3.5.

(5.10) Lemma

If $\underline{\underline{X}}$ is any class of primitive groups, then $\underline{\underline{X}} \subseteq p(\underline{\underline{X}}) \subseteq a(\underline{\underline{Y}})$ where $\underline{\underline{Y}}$ is the $\underline{D}$-class with boundary $b^*(\underline{\underline{X}})$.

Proof It is clear that $\underline{\underline{X}} \subseteq p(\underline{\underline{X}})$. Let G be a $p(\underline{\underline{X}})$-group with $K \in \mathrm{Stab}(G)$ and let J be a $\underline{\underline{Y}}$-projector of K. Let L be a (G,J)-pass group. Therefore $L \in p(\underline{\underline{X}})''$ and $L \in b^*(\underline{\underline{X}})$ since $L/F(L) \in Q(J) \subseteq \underline{\underline{Y}} = h(b^*(\underline{\underline{X}}))$. It follows from Lemma 2.24 that $G \in a(\underline{\underline{Y}})$. Hence $p(\underline{\underline{X}}) \subseteq a(\underline{\underline{Y}})$. $\square$

(5.11) Lemma

If $\underline{\underline{D}}$ is a $\underline{D}$-class and G is a primitive $\underline{\underline{D}}$-group satisfying $G \in b^*(G)$ and $b^*(G) \subseteq b(\underline{\underline{D}}) \cup (G)$, then $b(\underline{\underline{D}}) \subsetneq a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$ and hence $\underline{\underline{D}} >_{\neq}> h(b^*(b(\underline{\underline{D}}) \cup (G)))$.

Proof We suppose that $b(\underline{\underline{D}}) \not\subseteq a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$ and let H be a $b(\underline{\underline{D}})$-group of smallest order such that $H \notin a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$. Let $K \in \mathrm{Stab}(H)$ and let J be an $h(b^*(b(\underline{\underline{D}}) \cup (G)))$-projector of K.

We first consider the possibility that $J = K$. Then $H \in h(b^*(b(\underline{\underline{D}}) \cup (G))) \cup b^*(b(\underline{\underline{D}}) \cup (G))$ since $(Q\text{-}1)H \subseteq Q(J) \subseteq h(b^*(b(\underline{\underline{D}}) \cup (G)))$. However $H \notin a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$ and hence

$H \in h(b^*(b(\underline{\underline{D}}) \cup (G)))$. On the other hand $H \in b(\underline{\underline{D}}) \subseteq p(b(\underline{\underline{D}}))$ $\subseteq p(b(\underline{\underline{D}}) \cup (G))$ and therefore H has a quotient in $b^*(b(\underline{\underline{D}}) \cup (G))$. This contradiction allows us to take $|J| < |K|$. We show $J \in \underline{\underline{D}}$. For if not, then J has a quotient isomorphic with some $b(\underline{\underline{D}})$-group, B say. By the minimality of H , we have $B \in a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$. Therefore J has a quotient in $b^*(b(\underline{\underline{D}}) \cup (G))$. This contradicts the fact that $J \in h(b^*(b(\underline{\underline{D}}) \cup (G))$ and hence $J \in \underline{\underline{D}}$.

Now, by Theorem 5.3, all (H,J)-pass groups lie in $b(\underline{\underline{D}})$, and hence, by the minimality of H , all lie in $a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$. Now $J \in h(b^*(b(\underline{\underline{D}}) \cup (G)))$ and so all (H,J)-pass groups lie in $b^*(b(\underline{\underline{D}}) \cup (G))$. Therefore, by Lemma 2.24, we have $H \in a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$, our final contradiction. Thus $b(\underline{\underline{D}}) \subsetneq a(h(b^*(b(\underline{\underline{D}}) \cup (G))))$ and so $\underline{\underline{D}} >_{\neq}> h(b^*(b(\underline{\underline{D}}) \cup (G)))$ by Theorem 2.19. □

The construction $b^*(\ )$ gives rise to the following characterization of those Schunck classes which are $\underline{D}$-classes.

(5.12) <u>Theorem</u>

A Schunck class $\underline{\underline{H}}$ is a $\underline{D}$-class if and only if for all $a(\underline{\underline{H}})$-groups G , we have $b^*(G) \subseteq a(\underline{\underline{H}})$.

<u>Proof</u> If $\underline{\underline{H}}$ is a $\underline{D}$-class, it follows from Theorem 5.3 that $b^*(G) \subseteq p(G) \subseteq a(\underline{\underline{H}})$ for all groups $G \in a(\underline{\underline{H}})$.

Conversely, suppose that $b^*(G) \subseteq a(\underline{\underline{H}})$ for all $a(\underline{\underline{H}})$-groups G . Let G be a $b(\underline{\underline{H}})$-group and let H be any $(G,\underline{\underline{H}})$-pass group. Then

$H/F(H) \in \underline{\underline{H}}$ and $H \in p(G)$. Now H has no proper quotient in $b^*(G)$ since $b^*(G) \subseteq a(\underline{\underline{H}})$ and $H/F(H) \in \underline{\underline{H}}$. Therefore $H \in b(\underline{\underline{H}})$. Applying Theorem 5.2 completes the proof. □

We now consider the boundary of the meet of two $\underline{D}$-classes and obtain a result similar to Lemma 3.9(ii).

(5.13) Lemma

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes, then

$$b(\underline{\underline{X}} \wedge \underline{\underline{Y}}) = (b(\underline{\underline{X}}) \cap \underline{\underline{Y}}) \cup (\underline{\underline{X}} \cap b(\underline{\underline{Y}})) \cup (b(\underline{\underline{X}}) \cap b(\underline{\underline{Y}})) \quad .$$

Proof Let $G \in (b(\underline{\underline{Y}}) \cap b(\underline{\underline{X}})) \cup (b(\underline{\underline{Y}}) \cap \underline{\underline{X}})$. Therefore $G/F(G) \in \underline{\underline{X}} \cap \underline{\underline{Y}} = \underline{\underline{X}} \wedge \underline{\underline{Y}}$, since $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes. Now $G \notin \underline{\underline{Y}}$ and so $G \notin \underline{\underline{X}} \cap \underline{\underline{Y}} = \underline{\underline{X}} \wedge \underline{\underline{Y}}$. It follows therefore that $G \in b(\underline{\underline{X}} \wedge \underline{\underline{Y}})$. A similar argument applies if $G \in b(\underline{\underline{X}}) \cap \underline{\underline{Y}}$.

Suppose now that $G \in b(\underline{\underline{X}} \wedge \underline{\underline{Y}})$. Then $G \notin \underline{\underline{X}} \wedge \underline{\underline{Y}} = \underline{\underline{X}} \cap \underline{\underline{Y}}$. We assume that G is not an $\underline{\underline{X}}$-group. Then G has a quotient in $b(\underline{\underline{X}})$. Since $G/F(G) \in \underline{\underline{X}} \wedge \underline{\underline{Y}} << \underline{\underline{X}}$, we have $G \in b(\underline{\underline{X}})$. Furthermore $G \in b(\underline{\underline{Y}}) \cup \underline{\underline{Y}}$, since $G/F(G) \in \underline{\underline{Y}}$, and the result then follows. □

(5.14) Proposition

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes, then $b(\underline{\underline{X}} \wedge \underline{\underline{Y}}) = b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))$.

Proof Clearly $b(\underline{\underline{X}}) \subseteq p(b(\underline{\underline{X}})) \subseteq p(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))$. Then

$p(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})) \subseteq a(h(b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))))$ by Lemma 5.10. Therefore it follows from Theorem 2.19 that $\underline{\underline{X}} >> h(b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})))$. Similarly $\underline{\underline{Y}} >> h(b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})))$ and so $\underline{\underline{X}} \wedge \underline{\underline{Y}} >> h(b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))$.

Let $G \in b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))$. It is easy to see that $G \in b^*(b(\underline{\underline{X}})) \cup b^*(b(\underline{\underline{Y}}))$. Since $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes, we have $\underline{\underline{X}} = \underline{\underline{X}}^0$ and $\underline{\underline{Y}} = \underline{\underline{Y}}^0$ by Lemma 5.8. Therefore $G \in b(\underline{\underline{X}}^0) \cup b(\underline{\underline{Y}}^0)$ $= b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})$. Let $K \in \mathrm{Stab}(G)$ and $J \in \mathrm{Proj}_{\underline{\underline{X}} \wedge \underline{\underline{Y}}}(K)$. Let L be any (G,J)-pass group. Then $L/F(L) \in \underline{\underline{X}} \wedge \underline{\underline{Y}} = \underline{\underline{X}} \cap \underline{\underline{Y}}$ since $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes. However $G \in b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})$ and $\underline{\underline{X}}, \underline{\underline{Y}} \in \underline{D}$ together imply that $L \in a(\underline{\underline{X}}) \cup a(\underline{\underline{Y}})$ by Theorem 5.3. Therefore $L \in (b(\underline{\underline{X}}) \cap \underline{\underline{Y}}) \cup (b(\underline{\underline{X}}) \cap b(\underline{\underline{Y}})) \cup (b(\underline{\underline{Y}}) \cap \underline{\underline{X}})$ and hence $L \in b(\underline{\underline{X}} \wedge \underline{\underline{Y}})$ by Lemma 5.13. Therefore $G \in a(\underline{\underline{X}} \wedge \underline{\underline{Y}})$ by Lemma 2.24 and hence $b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})) \subseteq a(\underline{\underline{X}} \wedge \underline{\underline{Y}})$. Then $h(b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))) >> \underline{\underline{X}} \wedge \underline{\underline{Y}}$ and the proof is complete. □

The second construction plays a role similar to $\underline{\underline{X}}^{\prec}$ described earlier.

(5.15) <u>Definition</u>

Let $\underline{\underline{X}}$ be a class of primitive groups and set $\underline{\underline{X}}_0 = \underline{\underline{X}}$. For $i = 1,2,\ldots$ let n_i be the order of a smallest $\underline{\underline{X}}_{i-1}$-group G such that either (a) G has a proper quotient in $\underline{\underline{X}}_{i-1}$ or (b) some $(G,h(\underline{\underline{X}}_{i-1}))$-pass group does not lie in $\underline{\underline{X}}_{i-1}$. Remove from $\underline{\underline{X}}_{i-1}$ those groups of order n_i which satisfy either (a) or (b). Call the resulting class $\underline{\underline{X}}_i$. Now set $\underline{\underline{X}}_\infty = \bigcap_{i=0}^{\infty} \underline{\underline{X}}_i$.

(5.16) Lemma

If $\underline{\underline{X}}$ is a class of primitive groups, then $\underline{\underline{X}}_\infty$ is a $\underline{D}$-boundary.

Proof Since $\underline{\underline{X}}_\infty$ only contains primitive groups and no $\underline{\underline{X}}_\infty$-group has a proper quotient in $\underline{\underline{X}}_\infty$, it is clear that $\underline{\underline{X}}_\infty$ is a Schunck boundary. Let $G \in \underline{\underline{X}}_\infty$ and suppose some $(G,h(\underline{\underline{X}}_\infty))$-pass group does not lie in $\underline{\underline{X}}_\infty$. Then there is some $i \in \mathbb{N}$ for which $|G| = n_i$ and $G \in \underline{\underline{X}}_i$ but some $(G,h(\underline{\underline{X}}_i))$-pass group does not lie in $\underline{\underline{X}}_i$. But then $G \notin \underline{\underline{X}}_{i+1}$ and so $G \notin \underline{\underline{X}}_\infty$. This, together with Theorem 5.2, gives the required result. □

(5.17) Definition

For a Schunck class $\underline{\underline{X}}$ we define the $\underline{D}$-*closure* $\bar{\underline{\underline{X}}}$ of $\underline{\underline{X}}$ by $\bar{\underline{\underline{X}}} = h(a(\underline{\underline{X}})_\infty)$.

(5.18) Lemma

If $\underline{\underline{D}}$ is a $\underline{D}$-class, then $\bar{\underline{\underline{D}}} = \underline{\underline{D}}$.

Proof Let $\underline{\underline{D}}$ be a $\underline{D}$-class and let G be a $b(\underline{\underline{D}})$-group. By Theorem 5.3 we know that all G-pass groups lie in $a(\underline{\underline{D}})$ and G has no quotient in $b(\underline{\underline{D}})$ hence $G \in (a(\underline{\underline{D}}))_\infty = b(\bar{\underline{\underline{D}}})$. Therefore $b(\underline{\underline{D}}) \subseteq b(\bar{\underline{\underline{D}}}) \subseteq a(\underline{\underline{D}})$ and so $\underline{\underline{D}} = \bar{\underline{\underline{D}}}$ by Theorem 2.19. □

(5.19) Lemma

If $\underline{\underline{X}}$ is a Schunck class and $\underline{\underline{D}}$ is a $\underline{D}$-class strongly containing $\underline{\underline{X}}$, then $\underline{\underline{D}} >> \bar{\underline{\underline{X}}}$.

<u>Proof</u> Since $\underline{\underline{D}} >> \underline{\underline{X}}$, we have $a(\underline{\underline{D}}) \subseteq a(\underline{\underline{X}})$ by Theorem 2.19. If G is a minimal $a(\underline{\underline{D}})$-group then $G \cong Z_p$ for some prime p and so $G \in a(\underline{\underline{X}})_i$ for all i . Hence $G \in a(\underline{\underline{X}})_\infty = b(\overline{\underline{\underline{X}}}) \subseteq a(\overline{\underline{\underline{X}}})$. Now suppose that $G \in a(\underline{\underline{D}})$ and all smaller $a(\underline{\underline{D}})$-groups lie in $a(\overline{\underline{\underline{X}}})$. Let $K \in \mathrm{Stab}(G)$. We first show that $G \notin \overline{\underline{\underline{X}}}$.

Suppose, on the contrary, that $G \in \overline{\underline{\underline{X}}}$. Then $G \in a(\underline{\underline{X}}) \cap \overline{\underline{\underline{X}}}$ implies that there exists an i for which $G \in a(\underline{\underline{X}})_{i-1} \setminus a(\underline{\underline{X}})_i$. Thus $|G| = n_i$ and G satisfies either (a) or (b) of 5.15. Firstly suppose that G has a proper quotient in $a(\underline{\underline{X}})_{i-1}$. Then G has a proper quotient in $a(\underline{\underline{X}})_\infty = b(\overline{\underline{\underline{X}}})$ since $\{H \in a(\underline{\underline{X}})_{i-1} : |H| < n_i\} \subseteq a(\underline{\underline{X}})_\infty$. This contradicts $G \in \overline{\underline{\underline{X}}}$. Notice that the $h(a(\underline{\underline{X}})_{i-1})$-subgroups of K are precisely the $h(a(\underline{\underline{X}})_\infty)$-subgroups of K since $\{H \in b(h(a(\underline{\underline{X}})_{i-1})) : |H| < n_i\} = \{H \in b(h(a(\underline{\underline{X}})_\infty)) : |H| < n_i\}$ by the construction. Now all proper G-pass groups lie in $a(\underline{\underline{D}})$ since $\underline{\underline{D}} \in \underline{D}$, and hence all lie in $a(\overline{\underline{\underline{X}}})$ by the choice of G . In particular all $(G,h(a(\underline{\underline{X}})_{i-1}))$-pass groups lie in $b(\overline{\underline{\underline{X}}})$ and hence in $a(\underline{\underline{X}})_\infty \subseteq a(\underline{\underline{X}})_{i-1}$. Therefore (b) of 5.15 cannot hold and so we conclude that $G \notin \overline{\underline{\underline{X}}}$.

Now either $G \in b(\overline{\underline{\underline{X}}})$ or $K \notin \overline{\underline{\underline{X}}}$ and hence $L \lneqq K$. In the latter case we notice that all (G,L)-pass groups lie in $a(\underline{\underline{D}})$ since $\underline{\underline{D}}$ is a $\underline{D}$-class and hence in $a(\overline{\underline{\underline{X}}})$ by the choice of G . Then $G \in a(\overline{\underline{\underline{X}}})$ by Lemma 2.24. □

We now have as an immediate corollary.

(5.20) <u>Corollary</u>

If $\underline{\underline{X}}$ is a Schunck class, then $\overline{\underline{\underline{X}}}$ is the unique minimal $\underline{D}$-class strongly containing $\underline{\underline{X}}$. □

(5.21) Proposition

For $\underline{D}$-classes $\underline{\underline{X}}$ and $\underline{\underline{Y}}$, the boundary of $\underline{\underline{X}} \vee \underline{\underline{Y}}$ is given by $(a(\underline{\underline{X}}) \cap a(\underline{\underline{Y}}))_\infty$.

Proof Since $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are $\underline{D}$-classes, we have $\underline{\underline{X}} \vee \underline{\underline{Y}} \in \underline{D}$ and hence $b(\underline{\underline{X}} \vee \underline{\underline{Y}}) = (a(\underline{\underline{X}} \vee \underline{\underline{Y}}))_\infty$ by Lemma 5.18. Now Lemma 2.22 completes the proof. □

We now show, by the following example, that we cannot consider $h((b(\underline{\underline{X}}))_\infty)$ instead of $h((a(\underline{\underline{X}}))_\infty)$.

(5.22) Example

Let $\underline{\underline{X}}$ be the Schunck class with boundary $b(\underline{\underline{X}}) = (E(2/3)$, $E(2/5)$, $E(3/5)$, $Z_5)$. Let V be a faithful and irreducible $E(2/3)$-module over $GF(5)$. Let G be the semi-direct product of V with $E(2/3)$. Then $G \in a(\underline{\underline{X}}) \backslash b(\underline{\underline{X}})$. Notice that since $Z_3 \notin b(\underline{\underline{X}})$, we have $(b(\underline{\underline{X}}))_\infty = (b_5(\underline{\underline{X}}))_\infty$ and $(a(\underline{\underline{X}}))_\infty = (a_5(\underline{\underline{X}}))_\infty$. All proper G-pass groups lie in $b_5(\underline{\underline{X}})$ and hence no proper G-pass group has a proper quotient in $(a(\underline{\underline{X}}))_\infty$. Therefore $G \in (a(\underline{\underline{X}}))_\infty$ but $G \notin b(\underline{\underline{X}})$ implies $G \notin (b(\underline{\underline{X}}))_\infty$. □

Next we briefly consider sub-boundaries of $\underline{D}$-boundaries which are themselves $\underline{D}$-boundaries.

(5.23) Proposition

Let $\{f_1,\ldots,f_n\}$ be a set of functions, $f_i:\underline{\underline{S}} \to \mathbb{N}$, such that each f_i is constant on isomorphism classes and non-increasing on taking subgroups and quotients. Let $f_{n+1}(G) = |G|$ for all $G \in \underline{\underline{S}}$.

Suppose that $\underline{\underline{X}}$ and $\underline{\underline{D}}$ are $\underline{D}$-classes with $b(\underline{\underline{X}}) \subsetneq b(\underline{\underline{D}})$. Let $\underline{\underline{E}}_0$ be the class of stabilizers of groups in $b(\underline{\underline{D}})\backslash b(\underline{\underline{X}})$. For $i = 1,2,\ldots, n+1$ define a class $\underline{\underline{E}}_i$ by:

$$\underline{\underline{E}}_i = (G \in \underline{\underline{E}}_{i-1} : f_i(G) \le f_i(H) \text{ for all } H \in \underline{\underline{E}}_{i-1}) .$$

Suppose that $\underline{\underline{E}}_{n+1}$ is non-empty. Let $T \in \underline{\underline{E}}_{n+1}$ and choose a $(b(\underline{\underline{D}})\backslash b(\underline{\underline{X}}))$-group, H say, with stabilizer isomorphic with T. Then $b(\underline{\underline{X}}) \cup (H)$ is a $\underline{D}$-boundary.

Proof Since $b(\underline{\underline{X}}) \cup (H)$ is a subclass of a Schunck boundary $b(\underline{\underline{D}})$, it is itself a Schunck boundary, $b(\underline{\underline{Y}})$ say. To show that $\underline{\underline{Y}}$ is a $\underline{D}$-class we need to show that all $(G,\underline{\underline{Y}})$-pass groups lie in $b(\underline{\underline{Y}})$ for each $b(\underline{\underline{Y}})$-group G. Now $b(\underline{\underline{X}}) \subset b(\underline{\underline{Y}})$ implies that $\underline{\underline{X}} \gg \underline{\underline{Y}}$ by Theorem 2.19.

Let G be a $b(\underline{\underline{X}})$-group. Then, since $\underline{\underline{X}}$ is a $\underline{D}$-class, Theorem 5.2 applies to give that all $(G,\underline{\underline{X}})$-pass groups lie in $b(\underline{\underline{X}})$. Therefore all $(G,\underline{\underline{Y}})$-pass groups lie in $b(\underline{\underline{X}}) \subset b(\underline{\underline{Y}})$. Hence it only remains to show that all $(H,\underline{\underline{Y}})$-pass groups lie in $b(\underline{\underline{Y}})$. The result then follows from Theorem 5.2.

Let L be an $(H,\underline{\underline{Y}})$-pass group with $J \in \mathrm{Stab}(L)$. Then J is a quotient of a subgroup of T , and hence $f_i(J) \leq f_i(T)$ for $i = 1,2,\ldots, n+1$. We claim that $J \in \underline{\underline{D}}$. If not, then J has a quotient, J/N say, in $b(\underline{\underline{D}})\backslash b(\underline{\underline{Y}})$. Let $R \in \mathrm{Stab}(J/N)$. Then $f_i(R) \leq f_i(J/N) \leq f_i(J) \leq f_i(T)$ for $i = 1,2,\ldots,n+1$. Now $R \in \underline{\underline{E}}_0$ and so $R \in \underline{\underline{E}}_{n+1}$. Therefore since $T \in \underline{\underline{E}}_{n+1}$ we have $|T| = |R|$ and it follows that $|J| > |T|$. This contradicts $f_{n+1}(J) \leq f_{n+1}(T)$ and hence we may assume that $J \in \underline{\underline{D}}$. Now, by Theorem 5.3, all $(H,\underline{\underline{Y}})$-pass groups lie in $a(\underline{\underline{D}})$ and hence $L \in b(\underline{\underline{D}})$.

Suppose $L \in b(\underline{\underline{D}})\backslash b(\underline{\underline{Y}})$. Since $J \in \underline{\underline{E}}_0$ and $f_i(J) \leq f_i(T)$ for $i = 1,2,\ldots,n+1$, that T is an $\underline{\underline{E}}_{n+1}$-group implies $J \in \underline{\underline{E}}_{n+1}$ and so $|T| = |J|$. Therefore $J = T$ and hence $L = H \in b(\underline{\underline{Y}})$. This contradiction shows that all $(H,\underline{\underline{Y}})$-pass groups lie in $b(\underline{\underline{Y}})$ and the proof is complete. □

Examples of possible functions are $f_1(A) = \ell(A)$ and $f_2(A) = \gamma_q(A)$ for some fixed prime q , for each $A \in \underline{\underline{S}}$.

(5.24) Corollary

If $n \leq |b(\underline{\underline{D}})|$ for a $\underline{D}$-class $\underline{\underline{D}}$, then there is a $\underline{D}$-boundary $b(\underline{\underline{X}}_n)$ of size n contained in $b(\underline{\underline{D}})$.

Proof Take $\{f_1,\ldots,f_r\} = \phi$ in Proposition 5.23. Let $\underline{\underline{X}}_0 = \underline{\underline{S}}$.

Then $b(\underline{\underline{X}}_0) = \phi$. Applying Proposition 5.23 with $\underline{\underline{X}} = \underline{\underline{X}}_0$ yields a $\underline{D}$-boundary, $b(\underline{\underline{X}}_1)$ say, with $|b(\underline{\underline{X}}_1)| = 1$. Repeat the process with $\underline{\underline{X}} = \underline{\underline{X}}_i$ for $i = 1,\ldots,n-1$ to obtain $\underline{\underline{X}}_2,\ldots,\underline{\underline{X}}_n$. Then $b(\underline{\underline{X}}_n) \subseteq b(\underline{\underline{D}})$ and $|b(\underline{\underline{X}}_n)| = n$. □

(5.25) <u>Lemma</u>

Let $\underline{\underline{D}}$ be a $\underline{D}$-class and G a primitive $\underline{\underline{D}}$-group satisfying

(a) all proper $(G,\underline{\underline{D}})$-pass groups lie in $b(\underline{\underline{D}})$

(b) $b(\underline{\underline{D}}) \subseteq h(G)$.

Then $b(\underline{\underline{D}}) \cup (G)$ is a $\underline{D}$-boundary.

<u>Proof</u> Clearly, since $G \in \underline{\underline{D}}$ and $b(\underline{\underline{D}}) \subseteq h(G)$, the class $b(\underline{\underline{D}}) \cup (G)$ is a Schunck boundary, $b(\underline{\underline{H}})$ say. Now $b(\underline{\underline{D}}) \subset b(\underline{\underline{H}})$ implies that $\underline{\underline{D}} >> \underline{\underline{H}}$ by Theorem 2.19. Therefore all $(G,\underline{\underline{H}})$-pass groups lie in $b(\underline{\underline{D}}) \subset b(\underline{\underline{H}})$ using (a). Let B be a $b(\underline{\underline{D}})$-group and L a $(B,\underline{\underline{H}})$-pass group. Then L is a $(B,\underline{\underline{D}})$-pass group and so $L \in b(\underline{\underline{D}}) \subset b(\underline{\underline{H}})$ by Theorem 5.2. Now Theorem 5.2 can be applied again to give $\underline{\underline{H}} \in \underline{D}$. □

We now examine avoidance classes of $\underline{D}$-classes and see that not every infinite avoidance class of a $\underline{D}$-class contains an infinite $\underline{D}$-boundary. First we need a lemma.

(5.26) <u>Lemma</u>

If G lies in some $\underline{D}$-boundary, then $G \in b^*(G)$.

Proof Suppose $G \in b(\underline{\underline{X}})$ for some $\underline{D}$-class $\underline{\underline{X}}$. Since $G \in p(G)$, clearly G has a quotient, G/M say, isomorphic with some $b^*(G)$-group, L say. Since $L \in b^*(G)$, we have $L \in p(G)$ but then $L \in a(\underline{\underline{X}})$ by Theorem 5.3. Therefore L, and hence G, has a quotient in $b(\underline{\underline{X}})$. Since $b(\underline{\underline{X}})$ is a boundary, we have $G = L$ and hence $G \in b^*(G)$. □

Using this result we can construct examples of groups which cannot belong to any $\underline{D}$-boundary.

(5.27) Example

Let V be an irreducible $GF(2).S_3$-module faithful for S_3. Form the semi-direct product $G = [V]S_3$. Then Z_2 appears as a G-pass group (for example, as a $(G,\underline{\underline{I}})$-pass group). Therefore $Z_2 \in p(G)$ and it follows that $Z_2 \in b^*(G)$. Since G has a quotient isomorphic with Z_2 we have $G \notin b^*(G)$. Therefore we conclude from Lemma 5.26 that G does not lie in any $\underline{D}$-boundary. □

(5.28) Example

Let $\underline{\underline{D}}$ be the $\underline{D}$-class with boundary (A_4,Z_2) (that $\underline{\underline{D}}$ is a $\underline{D}$-class follows from Theorem 5.2).

Let H_n be the direct product of n-copies of S_3 and let V_n be a faithful and irreducible H_n-module over $GF(2)$. Let R_n be the semi-direct product $[V_n]H_n$. Then $R_n \in a(\underline{\underline{D}})$ for all $n \in \mathbb{N}$ and hence $a(\underline{\underline{D}})$ is infinite.

Take G to be any group in $a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. Then, by Lemma 4.1, we see that G cannot have a quotient isomorphic with A_4 . Therefore $Z_2 \in Q(G)$. Now $G \in a(\underline{\underline{D}}) = a_2(\underline{\underline{D}})$ and so Z_2 appears as a G-pass group. Therefore $G \notin b^*(G)$ and so no $\underline{D}$-boundary contains G by Lemma 5.26. Hence $b(\underline{\underline{D}})$ is the unique largest $\underline{D}$-boundary contained in $a(\underline{\underline{D}})$. In particular $a(\underline{\underline{D}})$ does not contain an infinite $\underline{D}$-boundary even though it is infinite. □

To complete this section we classify those $\underline{D}$-classes which complement a given $\underline{D}$-class in the $\underline{D}$-lattice. Later we shall see that such a classification of complements to a Schunck class in $\underline{H}$ is much more difficult.

(5.29) <u>Proposition</u>

Let $\underline{\underline{D}}$ be a $\underline{D}$-class and let $\sigma = \mathrm{char}(\underline{\underline{D}})$. Then a $\underline{D}$-class $\underline{\underline{D}}'$ complements $\underline{\underline{D}}$ in $\underline{D}$ if and only if $\underline{\underline{S}}_{\sigma'} << \underline{\underline{D}}' << \underline{\underline{Q}}^{\sigma}$. Hence $\underline{\underline{Q}}^{\sigma}$ is the unique maximal complement to $\underline{\underline{D}}$ and $\underline{\underline{S}}_{\sigma'}$ is the unique minimal complement.

<u>Proof</u> Let $\underline{\underline{D}}'$ complement $\underline{\underline{D}}$ in $\underline{D}$. We first establish $\underline{\underline{D}}' << \underline{\underline{Q}}^{\sigma}$. To do this, it is enough to show that $Z_p \in b(\underline{\underline{D}}')$ for all $p \in \sigma$. For then we have $(Z_p : p \in \sigma) = b(\underline{\underline{Q}}^{\sigma}) \subseteq a(\underline{\underline{D}}')$ and hence $\underline{\underline{D}}' << \underline{\underline{Q}}^{\sigma}$

by Theorem 2.19. Now $b(\underline{\underline{D}} \wedge \underline{\underline{D}}') = (Z_p : p \in \mathbb{P})$ since $\underline{\underline{D}} \wedge \underline{\underline{D}}' = \underline{\underline{I}}$. Therefore it follows from Lemma 5.13 that if $p \in \sigma$, then $Z_p \in b(\underline{\underline{D}}')$.

To show $\underline{\underline{D}}' \gg \underline{\underline{S}}_{\sigma'}$, we need $b(\underline{\underline{D}}') \subseteq a(\underline{\underline{S}}_{\sigma'})$ by Theorem 2.19. Therefore we must show that for each $b(\underline{\underline{D}}')$-group G any Hall σ'-subgroup of G (the $\underline{\underline{S}}_{\sigma'}$-projectors) avoids $F(G)$. It is enough to show that $F(G)$ is a σ-group. Let $F(G) = O_r(G)$ for some prime r . Now since $G \in b(\underline{\underline{D}}')$ we have $Z_r \in b(\underline{\underline{D}}')$ by Theorem 5.2 considering Z_r as a $(G,\underline{\underline{I}})$-pass group. Then $Z_r \notin b(\underline{\underline{D}})$ otherwise $Z_r \in b(\underline{\underline{D}}) \cap b(\underline{\underline{D}}') \subseteq a(\underline{\underline{D}} \vee \underline{\underline{D}}')$, but $\underline{\underline{D}} \vee \underline{\underline{D}}' = \underline{\underline{S}}$ and hence $a(\underline{\underline{D}} \vee \underline{\underline{D}}') = \phi$. Therefore $r \in \sigma$.

Conversely, we suppose that $\underline{\underline{D}}'$ is a $\underline{\underline{D}}$-class with $\underline{\underline{S}}_{\sigma'} \ll \underline{\underline{D}}' \ll \underline{\underline{Q}}^{\sigma}$. Let p be any prime. If $p \in \sigma'$, then $Z_p \in b(\underline{\underline{D}})$. If $p \in \sigma$, then $Z_p \in b(\underline{\underline{Q}}^{\sigma}) \subseteq a(\underline{\underline{D}}')$ and so $Z_p \in b(\underline{\underline{D}}')$. It follows from Lemma 5.13 that $\underline{\underline{D}} \wedge \underline{\underline{D}}' = \underline{\underline{I}}$. We show finally that $a(\underline{\underline{D}}) \cap a(\underline{\underline{D}}') = \phi$ and hence $\underline{\underline{D}} \vee \underline{\underline{D}}' = \underline{\underline{S}}$ by Lemma 2.22. If $G \in a(\underline{\underline{D}})$ and r is the prime divisor of $|F(G)|$, then $Z_r \in b(\underline{\underline{D}})$ by Theorem 5.3 and $r \in \sigma'$. However if $G \in a(\underline{\underline{D}}')$ we obtain $Z_r \in b(\underline{\underline{D}}') \subseteq a(\underline{\underline{S}}_{\sigma'})$. Therefore $Z_r \in b(\underline{\underline{S}}_{\sigma'})$ and so $r \notin \mathrm{char}(\underline{\underline{S}}_{\sigma'}) = \sigma'$. This shows that $a(\underline{\underline{D}}) \cap a(\underline{\underline{D}}') = \phi$. $\square$

(5.30) <u>Example</u>

We consider $\underline{\underline{N}}$, the class of nilpotent groups. We show that, for any prime p , the Schunck class $\underline{\underline{Q}}^p$ complements $\underline{\underline{N}}$ in $\underline{H}$. From Lemma 2.22, we have $a(\underline{\underline{N}} \vee \underline{\underline{Q}}^p) = a(\underline{\underline{N}}) \cap a(\underline{\underline{Q}}^p) = a(\underline{\underline{N}}) \cap (Z_p) = \phi$ and hence $\underline{\underline{N}} \vee \underline{\underline{Q}}^p = \underline{\underline{S}}$. Furthermore, by Lemma 3.9, $b(\underline{\underline{N}} \wedge \underline{\underline{Q}}^p) = b'(b(\underline{\underline{N}}) \cup (Z_p))$.

Now, for each prime q distinct from p, the group $E(p/q) \in b(\underline{\underline{N}})$. From the recipe for $b'(\ )$, we see that $Z_q \in b(\underline{\underline{N}} \wedge \underline{\underline{Q}}^p)$ for all primes q distinct from p. Clearly $Z_p \in b(\underline{\underline{N}} \wedge \underline{\underline{Q}}^p)$. Therefore $\underline{\underline{N}} \wedge \underline{\underline{Q}}^p = \underline{\underline{I}}$.

It is clear that the set of complements to $\underline{\underline{N}}$ in $\underline{H}$ does not have a unique maximal element since such a Schunck class must contain $\underline{\underline{Q}}^p$ for all primes p and therefore must be $\underline{\underline{S}}$, but $\underline{\underline{S}}$ does not complement $\underline{\underline{N}}$.

We now exhibit an ascending chain of complements to $\underline{\underline{N}}$ such that the union of Schunck classes in such a chain is itself not a complement.

Let $\underline{\underline{X}}_{\mathbb{P}}$ be a complement to $\underline{\underline{N}}$ constructed according to Försters recipe (2.α) (page 39). Then $a(\underline{\underline{X}}_{\mathbb{P}}) = b(\underline{\underline{X}}_{\mathbb{P}})$. Recall char $\underline{\underline{N}} = \mathbb{P}$. Let π be any non empty set of primes. Let $\underline{\underline{X}}_\pi$ be the Schunck class whose boundary consists of those $b(\underline{\underline{X}}_{\mathbb{P}})$-groups whose socles are π-groups. We claim $\underline{\underline{X}}_\pi$ complements $\underline{\underline{N}}$. Since $b(\underline{\underline{X}}_\pi) \subseteq b(\underline{\underline{X}}_{\mathbb{P}})$, we have $\underline{\underline{X}}_\pi >> \underline{\underline{X}}_{\mathbb{P}}$. Now $\underline{\underline{X}}_{\mathbb{P}} \vee \underline{\underline{N}} = \underline{\underline{S}}$ and so $\underline{\underline{X}}_\pi \vee \underline{\underline{N}} = \underline{\underline{S}}$. Let $p \in \pi$. Then, by the construction of the groups in $b(\underline{\underline{X}}_{\mathbb{P}})$, we see that $Z_p \in b(\underline{\underline{X}}_\pi \wedge \underline{\underline{N}})$. But $E(p/q) \in b(\underline{\underline{N}})$ for all primes q distinct from p and therefore $Z_q \in b(\underline{\underline{X}}_\pi \wedge \underline{\underline{N}})$ by Lemma 3.9. Hence $\underline{\underline{X}}_\pi \wedge \underline{\underline{N}} = \underline{\underline{I}}$.

Notice that if $\sigma \subset \pi$, then $b(\underline{\underline{X}}_\sigma) \subset b(\underline{\underline{X}}_\pi)$ and so $\underline{\underline{X}}_\sigma >> \underline{\underline{X}}_\pi$ by Theorem 2.19. Let $\mathbb{P} = \{p_1, p_2, \dots\}$ and consider $\underline{\underline{X}}_{\mathbb{P}} << \underline{\underline{X}}_{\mathbb{P}\setminus\{p_1\}} << \underline{\underline{X}}_{\mathbb{P}\setminus\{p_1,p_2\}} << \dots$. This is an ascending chain of complements to $\underline{\underline{N}}$ but $\bigvee_{r=1}^{\infty} \underline{\underline{X}}_{\mathbb{P}\setminus\{p_1,\dots,p_r\}} = \underline{\underline{S}}$ and $\underline{\underline{S}}$ is obviously not a complement to $\underline{\underline{N}}$. □

§6. Special $\underline{D}$-classes.

It has been conjectured that those Schunck classes $\underline{\underline{H}}$ which are n-maximal in $\underline{H}$ for some $n \in \mathbb{N}$ are precisely those for which $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$. It is easy to see that for a Schunck class, $\underline{\underline{H}}$, $a(\underline{\underline{H}}) = b(\underline{\underline{H}})$ if and only if for all Schunck classes $\underline{\underline{X}}$ strongly containing $\underline{\underline{H}}$ we have $b(\underline{\underline{X}}) \subseteq b(\underline{\underline{H}})$. We carry this idea over to $\underline{D}$, introducing special $\underline{D}$-classes and later we show that these $\underline{D}$-classes are precisely those which have constant depth in $\underline{D}$.

(6.1) Definition

We call a $\underline{D}$-class, $\underline{\underline{D}}$, *special* if, whenever $\underline{\underline{X}} \in \underline{D}$ and $\underline{\underline{X}} >> \underline{\underline{D}}$, we have $b(\underline{\underline{X}}) \subseteq b(\underline{\underline{D}})$.

(6.2) Lemma

Let $\underline{\underline{X}}$ and $\underline{\underline{D}}$ be $\underline{D}$-classes with $\underline{\underline{D}}$ special. If $\underline{\underline{X}} >> \underline{\underline{D}}$, then $\underline{\underline{X}}$ is special.

Proof Let $\underline{\underline{Y}}$ be a $\underline{D}$-class strongly containing $\underline{\underline{X}}$. Then $b(\underline{\underline{Y}}) \subseteq a(\underline{\underline{X}})$. Since $\underline{\underline{D}}$ is special, $b(\underline{\underline{X}}) \subseteq b(\underline{\underline{D}})$ and hence $b(\underline{\underline{X}}) \subseteq b(\underline{\underline{D}}) \cap a(\underline{\underline{X}})$. Now, if $G \in b(\underline{\underline{D}}) \cap a(\underline{\underline{X}})$, then G has a quotient in $b(\underline{\underline{X}}) \subseteq b(\underline{\underline{D}})$ but $b(\underline{\underline{D}})$ is a boundary so $G \in b(\underline{\underline{X}})$. Therefore $b(\underline{\underline{X}}) = b(\underline{\underline{D}}) \cap a(\underline{\underline{X}})$. Therefore $b(\underline{\underline{D}}) \cap b(\underline{\underline{Y}}) \subseteq b(\underline{\underline{X}})$. Now $\underline{\underline{Y}} >> \underline{\underline{D}}$ hence $b(\underline{\underline{Y}}) \subseteq b(\underline{\underline{D}})$. Therefore $b(\underline{\underline{Y}}) \subseteq b(\underline{\underline{X}})$ and $\underline{\underline{X}}$ is special. $\square$

(6.3) <u>Proposition</u>

If $\underline{\underline{D}}$ is a special $\underline{D}$-class, then $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p .

<u>Proof</u> Suppose $b(\underline{\underline{D}}) \neq b_p(\underline{\underline{D}})$ for any prime p . Choose distinct primes p and q such that $b_p(\underline{\underline{D}}) \neq \phi \neq b_q(\underline{\underline{D}})$. Then, by Theorem 5.2, we have $(Z_p,Z_q) \subseteq b(\underline{\underline{D}})$. Thus $E(q/p) \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. Let $b(\underline{\underline{X}}) = (E(q/p),Z_p)$. Then $\underline{\underline{X}}$ is a $\underline{D}$-class by Theorem 5.2. Furthermore $b(\underline{\underline{X}}) \subset a(\underline{\underline{D}})$ hence $\underline{\underline{X}} >> \underline{\underline{D}}$ but $b(\underline{\underline{X}}) \not\subseteq b(\underline{\underline{D}})$. This contradicts the assumption that $\underline{\underline{D}}$ is special. □

We now use an example of Förster to show that the converse of Proposition 6.3 is false.

(6.4) <u>Example</u>

(Förster [2], 9.6.) Let Q be the quaternion group of order 16 and D the dihedral group of order 8. Let $Z = Z(Q)$ and let U be a faithful, irreducible $GF(p),D$-module for $p \in \mathbb{P}\backslash\{2\}$. Now we may also consider U as a Q-module with $C_Q(U) = Z$. Let $K = [U]Q$ and $q \in P\backslash\{2,p\}$. Let V be a faithful, irreducible $GF(q).(K/Z)$-module. Consider V as a K-module with $C_K(V) = Z$ and form $H = [V].K$. Then $Soc(H) = Z \times V$ and so H has a faithful, irreducible module M over $GF(p)$. Define $G = [M]H$. We first show $G \in b^*(G)$. Clearly, it is enough to show that $[U]D \notin p(G)$.

Suppose that $[U]D$ is a (G,T)-pass group, $[V]T/C_T(V)$ say, for some $T \leq H$ and V a T-composition factor of $F(G)$. Let $S \in Syl_2(T)$. Suppose $|S| = 16$. Then $S \in Syl_2(G)$ and $Z \leq S$. Now $C_T(V) \triangleleft T$. If $[V]T/C_T(V) \cong [U]D$, then $2 \big| |C_T(V)|$. Therefore $1 \neq C_T(V) \cap S \triangleleft S$ and so $Z \leq C_T(V)$ because S is a 2-group. However, by Clifford's Theorem (Lemma 1.13), Z acts non-trivially on every Z-composition factor of $F(G)$. Therefore all (G,T)-pass groups have Sylow 2-subgroups of order 16 and hence none are isomorphic with $[U].D$. Therefore $|S| = 8$ but Q_{16} doesn't have a subgroup isomorphic with D_8 hence no (G,T)-pass group is isomorphic with $[U]D$.

Therefore $[U]D \notin p(G)$ and $G \in b^*(G)$. Let $b(\underline{\underline{D}}) = b^*(b^*(G) \cup ([U]D))$. Then $G \notin b(\underline{\underline{D}})$, but $G \in a(\underline{\underline{D}})$ by Lemma 5.10. It now follows by our next result, Theorem 6.5, that $\underline{\underline{D}}$ is not special. However, since $F(G) = O_p(G)$ and U is a p-group, we have $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$. $\square$

I think the following is probably about as useful a characterization of special $\underline{D}$-classes as we can hope for, even though it is not very efficient.

(6.5) Theorem

A $\underline{D}$-class $\underline{\underline{D}}$ is special if and only if, for all groups G in $a(\underline{\underline{D}}) \backslash b(\underline{\underline{D}})$, $G \notin b^*(G)$.

Proof Let $\underline{\underline{D}} \in \underline{D}$ be special. Suppose $G \in b^*(G)$ for some $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. By Theorem 5.3, $b^*(G) \subseteq p(G) \subseteq a(\underline{\underline{D}})$, hence $(h(G))^0 \gg \underline{\underline{D}}$. However, $b^*(G) \nsubseteq b(\underline{\underline{D}})$ contradicts $\underline{\underline{D}}$ being special.

Conversely, let $G \notin b^*(G)$ for all groups $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. Suppose $\underline{\underline{D}}$ is not special. Then there is a $\underline{D}$-class $\underline{\underline{X}}$ strongly containing $\underline{\underline{D}}$ but with $b(\underline{\underline{X}}) \nsubseteq b(\underline{\underline{D}})$. Since $b(\underline{\underline{X}}) \subseteq a(\underline{\underline{D}})$, there exists $K \in (a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})) \cap b(\underline{\underline{X}})$. It follows from Lemma 5.26 that $K \in b^*(K)$, since $K \in b(\underline{\underline{X}})$. This contradicts our hypotheses and hence concludes the proof. □

(6.6) Theorem

A $\underline{D}$-class $\underline{\underline{D}}$ is special if and only if, for all groups $G \in a(\underline{\underline{D}})$, $b^*(G) \subseteq b(\underline{\underline{D}})$.

Proof If $b^*(G) \subseteq b(\underline{\underline{D}})$ for all $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$, then $G \notin b^*(G)$, hence $\underline{\underline{D}}$ is special by Theorem 6.5.

Let $\underline{\underline{D}}$ be special and suppose that for some $G \in a(\underline{\underline{D}})$, $b^*(G) \nsubseteq b(\underline{\underline{D}})$. By Theorem 5.3, we have $b^*(G) \subseteq p(G) \subseteq a(\underline{\underline{D}})$. Therefore, there exists a group $H \in b^*(G) \cap (a(\underline{\underline{D}})\backslash b(\underline{\underline{D}}))$. Since $H \in b^*(G)$ and $(h(G))^0 \in \underline{D}$, we have, by Lemma 5.26, that $H \in b^*(H)$. Then $\underline{\underline{D}}$ is not special by Theorem 6.5. This contradiction gives the required result. □

(6.7) Remark

If $\underline{\underline{D}} \in \underline{D}$ and for some $H \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ there is a $b(\underline{\underline{D}})$-group,

G say, which satisfies $G \in Q(H) \cap p(H)$, then $H \notin b^*(H)$.

<u>Proof</u> Since $G \in p(H)$, G has a quotient in $b^*(H)$. Now H has a quotient isomorphic with G , hence H has a proper quotient in $b^*(H)$. Therefore $H \notin b^*(H)$. $\square$

The next result will be of use later in providing us with examples of $\underline{D}$-classes with constant depth n for any $n \in \mathbb{N}$.

(6.8) <u>Lemma</u>

If $G \in b^*(G)$ and $(h(G))^0$ is not special, then $h(b^*(G)\backslash(G))$ is not special.

<u>Proof</u> Let $\underline{\underline{D}} = (h(G))^0$ and suppose $\underline{\underline{D}}$ is not special. Then, by Theorem 6.5, there is a group $L \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ with $L \in b^*(L)$.

Suppose $G \in p(L)$. Let $T \in b^*(G)$. Then T has a quotient, T/N say, in $b^*(L) \subseteq a(\underline{\underline{D}})$. Then T/N has a quotient in $b(\underline{\underline{D}}) = b^*(G)$. Now $T \in b^*(G)$ and $b^*(G)$ is a boundary, hence $N = 1$ and $T \in b^*(L)$. Therefore $b^*(G) \subseteq b^*(L) \subseteq a(\underline{\underline{D}})$. It follows that $b^*(G) = b^*(L)$. Then we have a contradiction since $L \in b^*(L)$ but $L \notin b^*(G) = b(\underline{\underline{D}})$. Therefore $G \notin p(L)$.

Let $K \in Stab(L)$ and let T be an $h(b^*(G)\backslash(G))$-projector of K . If all (L,T)-pass groups lie in $b^*(G)$, then all lie in $b^*(G)\backslash(G)$, since $G \notin p(L)$, hence $L \in a(h(b^*(G)\backslash(G)))\backslash(b^*(G)\backslash(G))$ by Lemma 2.24. Therefore $h(b^*(G)\backslash(G))$ is not special.

If some (L,T)-pass group, M say, does not lie in $b^*(G)$, then $M \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ by Theorem 5.3. Thus, M has a quotient isomorphic with G, since $T \in h(b^*(G)\backslash(G))$. Now, by Lemma 4.1 some $b(\underline{\underline{D}})$-group has Sylow q-subgroups with class greater than those of G. However, $b(\underline{\underline{D}}) \leq p(G)$ hence all $b(\underline{\underline{D}})$-groups have Sylow q-subgroups with class at most equal to $\gamma_q(G)$. This contradiction completes the proof. $\square$

The next result provides us with examples of special $\underline{D}$-classes.

(6.9) Lemma

If $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$, for a $\underline{D}$-class $\underline{\underline{D}}$ and some prime p, and $c(\underline{\underline{D}}) \leq \underline{\underline{A}}$, then $\underline{\underline{D}}$ is special.

Proof Let $H \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ have a quotient isomorphic with $G \in b(\underline{\underline{D}})$. Let $K \in \mathrm{Stab}(G)$, hence $K \in \underline{\underline{A}}$. Let $L \in \mathrm{Stab}(H)$ and $D \in \mathrm{Proj}_{\underline{\underline{D}}}(L)$. Now $p \nmid |F(L)|$ since $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ and so $F(L) \leq D$. Suppose D is not nilpotent. Then we get an (H,D)-pass group with nilpotent length at least three. Since all $b(\underline{\underline{D}})$-groups have nilpotent length at most two, D is nilpotent. Similarly, if D is not abelian, then we get an (H,D)-pass group with non-abelian stabilizer contrary to our assumption $c(\underline{\underline{D}}) \leq \underline{\underline{A}}$.

We may therefore assume that D is abelian. Now $F(L) \leq D$ and D centralizes $F(L)$, hence $D \leq F(L)$ by Lemma 1.2. Therefore $D = F(L)$

and so $K = 1$. It follows that $G \cong Z_p$. Since $F(H) = O_p(H)$, Z_p appears as an H-pass group. We have, by Remark 6.7, that $H \notin b^*(H)$. Applying Theorem 6.5 completes the proof. □

(6.10) Lemma

Let $\underline{\underline{X}}$ be a $\underline{D}$-class with $Z_p \in b(\underline{\underline{X}})$ and $q \in char(\underline{\underline{X}})$ for primes p and q . Let V_n be a faithful, irreducible $GF(p).Z_{q^n}$-module for $n = 1,2,....$. If $G \in b_p(\underline{\underline{X}})$, then $(Q-1)(G) \cap \{[V_n].Z_{q^n} : n \in \mathbb{N}\} = \phi$.

Proof Suppose the result is false. Let $G \in b_p(\underline{\underline{X}})$ and let $[V_n].Z_{q^n}$ appear as a proper quotient of G for some $n \in \mathbb{N}$. Let $K \in Stab(G)$. Then K has a subgroup isomorphic with Z_{q^n} . Thus, by Theorem 5.2, $[V_n]Z_{q^n} \in b_p(\underline{\underline{X}})$. Since $b_p(\underline{\underline{X}})$ is a boundary, we have a contradiction of $G \in b_p(\underline{\underline{X}})$. □

It is now quite easy to see that any $\underline{D}$-class with at most three groups in its boundary and $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p , is special.

(6.11) Theorem

Let $\underline{\underline{D}}$ be a $\underline{D}$-class with $|b(\underline{\underline{D}})| \leq 3$. Then $\underline{\underline{D}}$ is special if and only if $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p . Moreover, if $\underline{\underline{D}}$ is special, $b(\underline{\underline{D}})$ has one of the following forms, for distinct primes p,q and r :

(1) $b(\underline{\underline{D}}) = (Z_p)$

(2) $b(\underline{\underline{D}}) = b^*(E(q/p)) = (E(q/p),Z_p)$.

(3) $b(\underline{\underline{D}}) = b^*(E(q/p),E(r/p)) = (E(q/p),E(r/p),Z_p)$

(4) $b(\underline{\underline{D}}) = b^*([V]\,Z_{q^2}) = ([V]\,Z_{q^2},E(q/p),Z_p)$, where V is a faithful, irreducible $GF(p).Z_{q^2}$-module;

(5) $b(\underline{\underline{D}}) = b^*(G) = (G,E(q/p),Z_p)$, where G is a primitive group with stabilizer an extraspecial q-group of exponent q and order q^3 , and $F(G) = O_p(G)$.

<u>Proof</u> Let $\underline{\underline{D}} \in \underline{D}$, $|b(\underline{\underline{D}})| \leq 3$ and $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p . Let $G \in b(\underline{\underline{D}})$. Then $G \in b^*(G)$ by Lemma 5.26. Suppose $\ell(G) \geq 3$. It is clear that only two primes divide $|G|$, p and q say. For, suppose p,q,r divide $|G|$, then $(G,E(q/p),E(r/p),Z_p) \subset b(\underline{\underline{D}})$ by Theorem 5.2. Next, let $K \in \mathrm{Stab}(G)$ and $Q \in \mathrm{Syl}_q(K)$. Now G cannot have a quotient isomorphic with Z_p since $G \in b^*(G)$ by Remark 6.7. Thus, $Q \nleq F(K)$, since $\ell(G) > 2$. However, $Z(Q)$ centralizes $F(K) = Oq(K)$, hence $Z(Q) \leq F(K)$ by Lemma 1.2. Therefore $\gamma_q(Q) \geq 2$. It follows that some (G,Q)-pass group, T say, has $\gamma_q(T) \geq 2$. Therefore $(G,T,E(q/p),Z_p) \subseteq b(\underline{\underline{D}})$. Again, this contradicts $|b(\underline{\underline{D}})| \leq 3$. Therefore $c(\underline{\underline{D}}) \subset \underline{\underline{N}}$.

The situation where $c(\underline{\underline{D}}) \subset \underline{\underline{A}}$ is covered by Lemma 6.9 and results in $b(\underline{\underline{D}})$ taking on one of the forms (1),(2),(3) or (4). We now assume $\gamma_q(K) > 1$ for some prime $q \neq p$. Then $\gamma_q(K) = 2$, otherwise some $(G,Z_2(O_q(K)))$-pass group, T say, has $\gamma_q(T) = 2$, but, by Theorem 5.2, $(G,T,E(q/p),Z_p) \subseteq b(\underline{\underline{D}})$ gives the usual contradiction.

Suppose there is a prime $r \notin \{q,p\}$ which divides $|K|$. Then $(G,E(r/p),E(q/p),Z_p) \subseteq b(\underline{\underline{D}})$. Therefore K is a q-group with class 2. If K has exponent greater than q, we have $(G,[V].Z_{q^2},E(q/p),Z_p) \subseteq b(\underline{\underline{D}})$. Therefore K has exponent q. Also, $Z(K)$ is cyclic by Corollary 1.12. Suppose now that $|K| > q^3$. Then, by Lemma 1.8, K has a subgroup, Y say, of order q^3 with $\gamma_q(Y) = 2$ and $Z(Y) = Z(K)$. Then some (G,Y)-pass group, T say, has $\gamma_q(T) = 2$, hence $(G,T,E(q/p),Z_p) \subseteq b(\underline{\underline{D}})$. We conclude that $|K| = q^3$, $Z(K) \cong Z_q$, hence K is extraspecial and we have form (5). It follows from Lemma 6.8 that, in this case, $\underline{\underline{D}}$ is special.

The converse follows immediately from Proposition 6.3. □

Recall, that if $b(\underline{\underline{H}})$ is a Schunck boundary and $G \in b(\underline{\underline{H}})$, then $b(\underline{\underline{H}})\backslash(G)$ is also a Schunck boundary. This is not necessarily true for $\underline{D}$-boundaries. We look at those $b(\underline{\underline{D}})$-groups, for $\underline{\underline{D}} \in \underline{D}$, for which $b(\underline{\underline{D}})\backslash(G)$ is a $\underline{D}$-boundary.

(6.12) Definition

For a $\underline{D}$-class $\underline{\underline{D}}$, define the *generating class* of $b(\underline{\underline{D}})$, denoted $g(\underline{\underline{D}})$, by:

$$g(\underline{\underline{D}}) = (G \in b(\underline{\underline{D}}) : G \notin p(H)\backslash(H) \text{ for all } H \in b(\underline{\underline{D}})) .$$

We call any element of $g(\underline{\underline{D}})$ a *generator*.

(6.13) Lemma

For a $\underline{D}$-class $\underline{\underline{D}}$ with finite boundary, $g(\underline{\underline{D}})$ is the unique class of $b(\underline{\underline{D}})$-groups, $\underline{\underline{R}}$, for which $b^*(\underline{\underline{R}}) = b(\underline{\underline{D}})$ and for any $R \in \underline{\underline{R}}$, $b^*(\underline{\underline{R}}\backslash(R)) \neq b(\underline{\underline{D}})$.

Proof Let $\underline{\underline{R}} \subseteq b(\underline{\underline{D}})$ satisfy $b^*(\underline{\underline{R}}) = b(\underline{\underline{D}})$ but $b^*(\underline{\underline{R}}\backslash R) \neq b(\underline{\underline{D}})$ for any $R \in \underline{\underline{R}}$. Let $G \in g(\underline{\underline{D}})$. Then, if $G \notin \underline{\underline{R}}$, we have $G \in p(\underline{\underline{R}})$, since $b^*(\underline{\underline{R}}) = b(\underline{\underline{D}})$. Therefore $G \in p(H)$ for some $H \in b(\underline{\underline{D}})\backslash(G)$ and hence $G \notin g(\underline{\underline{D}})$. Thus, we have shown $g(\underline{\underline{D}}) \subseteq \underline{\underline{R}}$. Suppose $\underline{\underline{R}} \neq g(\underline{\underline{D}})$. Let $S \in \underline{\underline{R}}\backslash g(\underline{\underline{D}})$. Since $S \notin g(\underline{\underline{D}})$, we have $S \in p(T)\backslash(T)$ for some $T \in b(\underline{\underline{D}})$. We may assume $T \in g(\underline{\underline{D}})$, otherwise we can replace S by T and repeat the argument. Therefore $b^*(\underline{\underline{R}}\backslash(S)) = b(\underline{\underline{D}})$, contrary to the restrictions imposed on $\underline{\underline{R}}$, hence $g(\underline{\underline{D}}) = \underline{\underline{R}}$. □

(6.14) Definition

Let $G \in g(\underline{\underline{D}})$ for a $\underline{D}$-class $\underline{\underline{D}}$. Call G a *type I generator* if $b(\underline{\underline{D}})\backslash(G)$ is a $\underline{D}$-boundary, and a *type II generator* otherwise. Notice that type I generators always exist if $|b(\underline{\underline{D}})|$ is finite by Proposition 5.23. However $(Z_p, E(q/p), E(q^2/p), \ldots\ldots)$ is a $\underline{D}$-boundary with no generators at all.

(6.15) Example

Let G be a polyprimitive group of type $(2,3,5,7)$ and

$H = E(3/5)$. Define a $\underline{D}$-class $\underline{\underline{D}}$ by $b(\underline{\underline{D}}) = b^*(G,H)$. It is clear that each of G and H lies in $b(\underline{\underline{D}})$ and that each is in $g(\underline{\underline{D}})$. Since G is the unique group having maximal order in $b(\underline{\underline{D}})$, we have, by Proposition 5.23, that G is a type I generator. Now G has as a G-pass group a polyprimitive group of type $(3,5,7)$, L say. Since L has a quotient isomorphic with H , we have $L \notin b(\underline{\underline{D}})$. However, L has no quotient in $b(\underline{\underline{D}})\backslash(H)$, hence, if $b(\underline{\underline{D}})\backslash(H)$ is a $\underline{D}$-boundary, we have $L \in b(\underline{\underline{D}})\backslash(H)$, by Theorem 5.3. This contradiction shows H to be a type II generator. □

(6.16) Lemma

If G is a type II generator for a $\underline{D}$-boundary $b(\underline{\underline{D}})$, then there is a $b(\underline{\underline{D}})$-group, H , such that $G \in Q(T)$ for some $T \leq S \in \mathrm{Stab}(H)$.

Proof Since G is a type II generator, $b(\underline{\underline{D}})\backslash(G)$ is not a $\underline{D}$-boundary, however, it is a Schunck boundary, $b(\underline{\underline{H}})$ say. Since $\underline{\underline{H}} \nsubseteq \underline{\underline{D}}$, we conclude, from Theorem 5.2, that for some $b(\underline{\underline{H}})$-group, L say, not all $(L,\underline{\underline{H}})$-pass groups lie in $b(\underline{\underline{H}})$. Therefore, since G is a generator, there is a subgroup, M , of $N \in \mathrm{Stab}(L)$ such that $M \in \underline{\underline{H}}\backslash\underline{\underline{D}}$. Then M has a quotient isomorphic with G . □

(6.17) Lemma

If $\underline{\underline{D}}$ is a special $\underline{D}$-class, then $b(\underline{\underline{D}})$ only has type I generators.

<u>Proof</u> Suppose G is a type II generator for $b(\underline{\underline{D}})$. Let $\underline{\underline{H}}$ be the Schunck class with boundary $b(\underline{\underline{D}})\backslash(G)$. Then $\underline{\underline{H}}$ is not a $\underline{D}$-class. Therefore, by Theorem 5.2 there is a $b(\underline{\underline{H}})$-group, H say, with not all $(H,\underline{\underline{H}})$-pass groups lying in $b(\underline{\underline{H}})$. Let R be such a pass group of smallest order and let $S \in \mathrm{Stab}(R)$. Then $R \in \underline{\underline{H}}$. Now $R \in a(\underline{\underline{D}})$ by Theorem 5.3 and $R \notin b(\underline{\underline{D}})\backslash(G)$. Since G is a generator, $R \neq G$. Hence $R \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. Therefore $S \in \underline{\underline{H}}\backslash\underline{\underline{D}}$ and so $G \in Q(S)$.

Now $R \in p(H)$ implies that R has a quotient $R/N \in b^*(H) \subseteq a(\underline{\underline{D}})$. Suppose $N > 1$. Now $R/N \in b^*(H) \subseteq p(H)$ and so R/N is an (H,J)-pass group for some subgroup J of a stabilizer of H. However $R/N \in \underline{\underline{H}}$ and so R/N is an (H,X)-pass group where $X \in \mathrm{Proj}_{\underline{\underline{H}}}(J)$. Thus R/N is an $(H,\underline{\underline{H}})$-pass group. By choice of R we have $R/N \in b(\underline{\underline{H}})$. This contradiction yields $N = 1$ and $R \in b^*(H)$. It follows from Lemma 5.26 that $R \in b^*(R)$. Now $R \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ yields that $\underline{\underline{D}}$ is not special by Theorem 6.5. Therefore $b(\underline{\underline{D}})$ has no type II generators. □

We now lay some foundations for our main result concerning $\underline{D}$-classes with constant depth. Recall that a $\underline{D}$-class $\underline{\underline{D}}$ has constant depth n if each chain of $\underline{D}$-classes from $\underline{\underline{D}}$ to $\underline{\underline{S}}$ can be refined to give one of length n .

(6.18) <u>Lemma</u>

If $\underline{\underline{D}}$ is a $\underline{D}$-class with constant depth n , then $|b(\underline{\underline{D}})| \leq n$.

Proof We suppose on the contrary that $|b(\underline{\underline{D}})| > n$. By Corollary 5.24, there is a $\underline{D}$-boundary, $b(\underline{\underline{X}}_n)$ say, of size n contained in $b(\underline{\underline{D}})$. Similarly there is a $\underline{D}$-boundary, $b(\underline{\underline{X}}_{n-1})$ say, of size $n-1$ contained in $b(\underline{\underline{X}}_n)$. Repeating this provides us with a proper chain of $\underline{D}$-boundaries $\phi = b(\underline{\underline{X}}_0) \subset b(\underline{\underline{X}}_1) \subset \ldots \subset b(\underline{\underline{X}}_n) \subset b(\underline{\underline{D}})$. This yields the corresponding proper chain of $\underline{D}$-classes $\underline{\underline{S}} >> \underline{\underline{X}}_1 >> \ldots\ldots >> \underline{\underline{X}}_n >> \underline{\underline{D}}$ with length $n+1$. This contradicts the hypothesis that $\underline{\underline{D}}$ has constant depth n and so we conclude that $|b(\underline{\underline{D}})| \leq n$. □

We see later that in fact we must have equality here, but this requires much more work.

(6.19) Lemma

Let $\underline{\underline{J}}$ and $\underline{\underline{D}}$ be $\underline{D}$-classes with $b(\underline{\underline{J}}) \subsetneq b(\underline{\underline{D}})$ and such that $\underline{\underline{J}}$ does not have constant depth j for any $j \leq n-1$. Then $\underline{\underline{D}}$ cannot have constant depth n .

Proof Suppose $\underline{\underline{D}}$ has constant depth n . Since $b(\underline{\underline{J}}) \subset b(\underline{\underline{D}})$, there is a $\underline{D}$-maximal chain C from $\underline{\underline{D}}$ to $\underline{\underline{J}}$. Because $\underline{\underline{J}}$ does not have constant depth j for any $j \leq n-1$, either there are $\underline{D}$-chains from $\underline{\underline{J}}$ to $\underline{\underline{S}}$ with length at least n and so, when taken with C , these give rise to $\underline{D}$-chains from $\underline{\underline{D}}$ to $\underline{\underline{S}}$ with length at least $n+1$, or there are $\underline{D}$-maximal chains of differing lengths from $\underline{\underline{J}}$ to $\underline{\underline{S}}$. In the second case, such chains joined with C yield $\underline{D}$-maximal chains from

$\mathbf{D}$ to $\mathbf{S}$ with differing lengths. In either case therefore we have a contradiction of $\mathbf{D}$ having constant depth n . □

Our next aim is to obtain a characterization of meet-irreducible $\underline{D}$-classes similar to that for Schunck classes in Proposition 3.10.

(6.20) Proposition

Let $\mathbf{D}$ be a $\underline{D}$-class with finite boundary. Then $\mathbf{D}$ is meet-irreducible in $\underline{D}$ if and only if $|g(\mathbf{D})| = 1$ or $\mathbf{D} = \mathbf{S}$.

Proof Let T be a type I generator for $b(\mathbf{D})$. Define a $\underline{D}$-class by $b(\mathbf{X}) = b(\mathbf{D})\backslash(T)$. Since $T \in b(\mathbf{D})$, we have $T \in b^*(T)$ by Lemma 5.26. Let $\mathbf{T} = (h(T))^0$. We show $\mathbf{D} = \mathbf{T} \wedge \mathbf{X}$.

Now $b(\mathbf{T} \wedge \mathbf{X}) = b^*(b^*(T) \cup b(\mathbf{X}))$ by Proposition 5.14. Then $b(\mathbf{D}) = b(\mathbf{X}) \cup (T) \subseteq b(\mathbf{X}) \cup b^*(T) \subseteq a(\mathbf{T} \wedge \mathbf{X})$ by Lemma 5.10, hence $\mathbf{D} >> \mathbf{T} \wedge \mathbf{X}$. However $b^*(T) \subseteq a(\mathbf{D})$ by Theorem 5.3 and so $\mathbf{T} >> \mathbf{D}$. Clearly $\mathbf{X} >> \mathbf{D}$ and hence $\mathbf{T} \wedge \mathbf{X} >> \mathbf{D}$. Therefore $\mathbf{D} = \mathbf{T} \wedge \mathbf{X}$.

Now $T \notin a(\mathbf{X})$ since $b(\mathbf{D})$ is a boundary and so T has no quotient in $b(\mathbf{X})$. Therefore $\mathbf{T} >|> \mathbf{X}$.

Suppose $|g(\mathbf{D})| > 1$ and let $B \in g(\mathbf{D})\backslash(T)$. We show $B \notin a(\mathbf{T})$. If $B \in a(\mathbf{T})$, then B has a quotient in $b^*(T) \subseteq a(\mathbf{D})$. Now $B \in b(\mathbf{D})$ yields $B \in b^*(T)$ and so, in particular, we have $B \in p(T)$. This contradicts $B \in g(\mathbf{D})$, hence $B \notin a(\mathbf{T})$. Therefore $\mathbf{X} >|> \mathbf{T}$ and so $\mathbf{D}$ is meet-reducible.

Suppose now that $|g(\underline{\underline{D}})| = 1$. Then $b(\underline{\underline{D}}) = b^*(T)$ for $T \in g(\underline{\underline{D}})$ by Lemma 6.13. We assume $\underline{\underline{D}}$ is meet-reducible and obtain a contradiction. Let $\underline{\underline{D}} = \underline{\underline{X}} \wedge \underline{\underline{Y}}$ for $\underline{D}$-classes $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ each distinct from $\underline{\underline{D}}$. Then $b(\underline{\underline{D}}) = b^*(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))$ by Proposition 5.14. Now $T \in b(\underline{\underline{D}})$ implies that $T \in p(b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}}))$. Let $G \in b(\underline{\underline{X}}) \cup b(\underline{\underline{Y}})$ be such that $T \in p(G)$. Without loss of generality we assume that G is a $b(\underline{\underline{X}})$-group. Then $G \in b^*(G)$ by Lemma 5.26 and $(h(G))^0 >> \underline{\underline{X}}$ by Proposition 5.9. Now $T \in p(G)$ and so $b(\underline{\underline{D}}) = b^*(T) \subseteq p(T) \subseteq p(G) \subseteq a(h(G)^0)$ by Lemma 5.10. Therefore $\underline{\underline{D}} >> (h(G))^0 >> \underline{\underline{X}}$. However $\underline{\underline{X}} >> \underline{\underline{D}}$ and so $\underline{\underline{X}} = \underline{\underline{D}}$. This contradiction completes the proof. □

Notice that, from Lemma 2.37 and Proposition 3.10, we have:

A Schunck class $\underline{\underline{H}}$ is uniquely determined by the meet-irreducible Schunck classes strongly containing it.

We now produce the analogous result for $\underline{D}$-classes.

(6.21) <u>Proposition</u>

A $\underline{D}$-class $\underline{\underline{D}}$ is uniquely determined by the meet irreducible $\underline{D}$-classes strongly containing it.

<u>Proof</u> Let $b(\underline{\underline{D}}) = (B_1, B_2, \ldots\ldots)$. Since $B_i \in b(\underline{\underline{D}})$, we have $B_i \in b^*(B_i)$ by Lemma 5.26. Therefore $g(h(B_i)^0) = (B_i)$. Hence $h(B_i)^0$ is a meet irreducible $\underline{D}$-class by Proposition 6.20. Since $B_i \in b(\underline{\underline{D}})$, it follows that $h(B_i) >> \underline{\underline{D}}$. Now Proposition 5.9 yields $h(B_i)^0 >> \underline{\underline{D}}$. Thus $\wedge_i h(B_i)^0 >> \underline{\underline{D}}$. Let G be an $\wedge_i h(B_i)^0$-group. Then $G \in h(B_i)$ for each i since $h(B_i) >> h(B_i)^0 >> \wedge_j h(B_j)^0$. Therefore G has no quotient in $b(\underline{\underline{D}})$ and so is a $\underline{\underline{D}}$-group. Thus $\underline{\underline{D}} = \wedge_i h(B_i)^0$. □

§7. Single chained $\underline{D}$-classes.

(7.1) Definition

A $\underline{D}$-class $\underline{\underline{D}}$ will be called *single chained* if there is only one chain of $\underline{D}$-classes from $\underline{\underline{D}}$ to $\underline{\underline{S}}$ which is maximally refined.

We obtain a boundary characterization of these $\underline{D}$-classes and then give examples of such classes with constant depth n for any $n \in \mathbb{N}$. First we need two results which will play a key part in the proof of the main theorem in §8.

(7.2) Lemma

If $\underline{\underline{D}}$ is a special $\underline{D}$-class with $|b(\underline{\underline{D}})| = n$, then $\underline{\underline{D}}$ has constant depth n.

Proof We use induction on the size of the boundary of $\underline{\underline{D}}$. If $|b(\underline{\underline{D}})| = 1$ and $\underline{\underline{D}}$ is special, it is clear that $\underline{\underline{D}}$ has constant depth one.

Let $|b(\underline{\underline{D}})| = n$ and assume the result is true for all special $\underline{D}$-classes with boundaries of size less than n. By Corollary 5.24, there is a $\underline{D}$-class $\underline{\underline{X}}$ with $\underline{\underline{X}} >> \underline{\underline{D}}$ and $|b(\underline{\underline{X}})| = n-1$. Now $\underline{\underline{D}}$ is special and hence $\underline{\underline{X}}$ is special by Lemma 6.2. Therefore by our induction hypothesis $\underline{\underline{X}}$ has constant depth $n-1$.

Suppose $\underline{\underline{X}} >> \underline{\underline{Y}} >> \underline{\underline{D}}$ for $\underline{\underline{Y}} \in \underline{D}$. Then $\underline{\underline{Y}}$ is special by Lemma 6.2. Since $|b(\underline{\underline{X}})| = n-1$ and $|b(\underline{\underline{D}})| = n$, we must have $\underline{\underline{Y}} = \underline{\underline{X}}$ or

$\underline{\underline{Y}} = \underline{\underline{D}}$ because $b(\underline{\underline{X}}) \subseteq b(\underline{\underline{Y}}) \subseteq b(\underline{\underline{D}})$. Therefore $\underline{\underline{D}}$ is $\underline{D}$-maximal in $\underline{\underline{X}}$ and hence there is a maximal refined chain of $\underline{D}$-classes from $\underline{\underline{D}}$ to $\underline{\underline{S}}$ of length n .

It is clear that all $\underline{D}$-maximal $\underline{D}$-chains have lengths at most n , since $|b(\underline{\underline{D}})| = n$ and $\underline{D}$ is special. Suppose one such chain has length less than n , $\underline{\underline{D}} \ll \underline{\underline{H}}_1 \ll \ldots\ldots \ll \underline{\underline{H}}_t = \underline{\underline{S}}$ say, with $t < n$. Then we may assume $|b(\underline{\underline{H}}_1)| = n-1$ by Proposition 5.23. Now $\underline{\underline{H}}_1$ is special by Lemma 6.2, and so $\underline{\underline{H}}_1$ has constant depth $n-1$ by our induction hypothesis. Therefore $\underline{\underline{H}}_1 \ll \ldots\ldots \ll \underline{\underline{H}}_t = \underline{\underline{S}}$ is not a $\underline{D}$-maximal chain. Therefore all $\underline{D}$-maximal $\underline{D}$-chains from $\underline{\underline{D}}$ to $\underline{\underline{S}}$ have length n and hence $\underline{\underline{D}}$ has constant depth n . □

(7.3) Proposition

If $\underline{\underline{D}}$ be a $\underline{D}$-class with constant depth n and $|b(\underline{\underline{D}})| = n$, then $\underline{\underline{D}}$ is special.

Proof Suppose $\underline{\underline{D}}$ is not special. Then there is a group G in $a(\underline{\underline{D}}) \backslash b(\underline{\underline{D}})$ with $G \in b^*(G) \subseteq a(\underline{\underline{D}})$, by Theorem 6.5. Take such a G of least order. Let $H \in b^*(G)$. Then, if $H \neq G$, we have $|H| < |G|$. Theorem 5.3 shows $H \in a(\underline{\underline{D}})$ and since $H \in b^*(G)$, it follows from Lemma 5.26 that $H \in b^*(H)$. Now minimality of $|G|$ yields $H \in b(\underline{\underline{D}})$ and hence $b^*(G)\backslash(G) \subseteq b(\underline{\underline{D}})$. By Proposition 5.23, $b^*(G)\backslash(G)$ is a $\underline{D}$-boundary since G is the unique group having maximal order in $b^*(G)$. Let $b(\underline{\underline{D}}) \backslash (b^*(G)\backslash(G)) = (B_1,\ldots,B_t)$ with $|B_1| \leq |B_2| \leq \ldots\ldots \leq |B_t|$.

Then, by Lemma 5.25, $(b^*(G)\backslash(G)) \cup (B_1,\ldots,B_i)$ is a $\underline{D}$-boundary, $b(\underline{\underline{H}}_i)$ say, for $i = 1,\ldots,t$. If $b(\underline{\underline{H}}_{i-1}) \cup (G)$ is a $\underline{D}$-boundary and G has no quotient isomorphic with B_i, then $b(\underline{\underline{H}}_i) \cup (G)$ is a $\underline{D}$-boundary. Choose i such that G has a quotient isomorphic with B_{i+1} but $Q(G) \cap (B_1,\ldots,B_i) = \phi$. Then $G \notin \underline{\underline{H}}_{i+1}$ but $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. Therefore, if $K \in \mathrm{Stab}(G)$, then K has a quotient isomorphic with B_{i+1}. Let $T \in \mathrm{Proj}_{\underline{\underline{H}}_{i+1}}(K)$. Then $|T| < |K|$.

Let R be a (G,T)-pass group. Then R has a quotient in $b^*(G)$ but $T \in \underline{\underline{H}}_{i+1}$ and so T has no quotient in $b^*(G)\backslash(G) \subseteq b(\underline{\underline{H}}_{i+1})$. Furthermore, since $|T| < |G|$, $Q(T) \cap b^*(G) = \phi$ and hence $R \in b^*(G)$. Now $|R| < |G|$ hence $R \in b(\underline{\underline{H}}_{i+1})$. Therefore $G \in a(\underline{\underline{H}}_{i+1})$ by Lemma 2.24. Define $\underline{\underline{H}}_j^G$ by $b(\underline{\underline{H}}_j^G) = b(\underline{\underline{H}}_j) \cup (G)$ for $j = 1,\ldots,i$, so $\underline{\underline{H}}_j^G \in \underline{D}$. Since $b(\underline{\underline{H}}_i^G) \subsetneq a(\underline{\underline{H}}_{i+1})$ we have $\underline{\underline{H}}_{i+1} \lneq \underline{\underline{H}}_i^G$. We now have a proper $\underline{D}$-chain of length $t+1$:

$$h(b^*(G)\backslash(G)) \gneq (h(G))^0 \gneq \underline{\underline{H}}_1^G \gneq \cdots \gneq \underline{\underline{H}}_i^G \gneq \underline{\underline{H}}_{i+1} \gneq \cdots \gneq \underline{\underline{H}}_t = \underline{\underline{D}} .$$

Now $|b^*(G)\backslash(G))| = n-t$ and so, by Corollary 5.24, there is a proper $\underline{D}$-chain of length $n-t$ from $h(b^*(G)\backslash(G))$ to $\underline{\underline{S}}$. Taken together with the above chain we obtain a $\underline{D}$-chain of length $n+1$ from $\underline{\underline{D}}$ to $\underline{\underline{S}}$. This contradicts our hypothesis that $\underline{\underline{D}}$ has constant depth n. $\square$

(7.4) Theorem

A $\underline{D}$-class $\underline{\underline{D}}$ is single chained if and only if $b(\underline{\underline{D}}) = (B_1,\ldots,B_n)$ and $b^*(B_i) = (B_i,B_{i+1}, \ldots , B_n)$ for $i = 1,2,\ldots,n$ where $|B_1| \geq |B_2| \geq \ldots\ldots \geq |B_n|$.

Proof Let $b(\underline{\underline{D}})$ be as described. The result is clearly true for $n = 1$. We use induction on the size of $b(\underline{\underline{D}})$. Assume the result is true for all $\underline{D}$-classes of this form with at most $n-1$ elements in their boundaries. Therefore $h(b^*(B_2))$ is single chained with constant depth $n-1$. By Proposition 7.3 $h(b^*(B_2))$ is special. Then $\underline{\underline{D}}$ is special by Lemma 6.8. Now $|b(\underline{\underline{D}})| = n$, so $\underline{\underline{D}}$ has constant depth n by Lemma 7.2. Since $\underline{\underline{D}}$ is special, if $\underline{\underline{D}}$ is $\underline{D}$-maximal in a $\underline{D}$-class $\underline{\underline{X}}$, then $b(\underline{\underline{X}}) = b(\underline{\underline{D}})\backslash(R)$ for some type I generator R . Suppose $R \neq B_1$. Then $B_1 \in b(\underline{\underline{X}})$ and $\underline{\underline{X}} \in \underline{D}$ so $b^*(B_1) \subseteq a(\underline{\underline{X}})$ by Lemma 5.7. Therefore $b(\underline{\underline{D}}) \subseteq a(\underline{\underline{X}}) \subseteq a(\underline{\underline{D}})$ and so $\underline{\underline{X}} = \underline{\underline{D}}$. This contradiction shows $\underline{\underline{D}}$ is single chained.

Conversely, let $\underline{\underline{D}}$ be single chained with constant depth n . The result is true for $n = 1$. We use induction on the depth of $\underline{\underline{D}}$ to prove the result. Since $\underline{\underline{D}}$ is single chained $b(\underline{\underline{D}})$ can only have one type I generator, B_1 say. Let $\underline{\underline{D}}_1 = h(b(\underline{\underline{D}}) \setminus (B_1))$. $\underline{\underline{D}}_1$ is single chained since $\underline{\underline{D}}$ is, and therefore $b(\underline{\underline{D}}_1)$ has at most one type I generator, B_2 say. By induction $b(\underline{\underline{D}}_1) = (B_2,\ldots\ldots,B_{t+1})$ where $\underline{\underline{D}}_1$ has depth $t \leq n-1$ and $b^*(B_i) = (B_i,B_{i+1},\ldots\ldots,B_{t+1})$ for $i = 2,3,\ldots,t+1$. Now, $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p , otherwise $(Z_p,Z_q) \subset b(\underline{\underline{D}})$ by

Theorem 5.2 for distinct primes p and q but then $\underline{\underline{S}} >> \underline{\underline{Q}}^p >> \underline{\underline{D}}$ and $\underline{\underline{S}} >> \underline{\underline{Q}}^q >> \underline{\underline{D}}$ cannot be refined to the unique maximally refined $\underline{D}$-chain from $\underline{\underline{D}}$ to $\underline{\underline{S}}$.

Suppose $\underline{\underline{D}}$ has a type II generator, B_ℓ say. Then B_ℓ is a generator for $\underline{\underline{D}}_1$. However, $\underline{\underline{D}}_1$ has just one generator B_2 . Therefore $B_\ell = B_2$. Now $b^*(B_2) = b(\underline{\underline{D}}_1) = b(\underline{\underline{D}})\backslash(B_1)$ hence $|B_2| \geq |B_i|$ for $i = 2,3,\ldots,t+1$. By Lemma 6.16, there is a subgroup S of a stabilizer K_1 of B_1 having a quotient isomorphic with B_2 . Let $T = F(K_1).S$. Then $F(K_1) \leq F(T)$. Now $O_p(T)$ centralizes $O_{p'}(F(T))$ and hence $F(K_1)$ because $p \nmid |F(K_1)|$. Therefore $O_p(T) \leq C_{K_1}(F(K_1)) \leq F(K_1) = O_{p'}(F(K_1))$, hence $O_p(T) = 1$. Therefore $T/F(T)$ has a quotient isomorphic with B_2 . By Lemma 1.29 there is a (B_1,T)-pass group, A say, with Sylow q-subgroups of greater class than those of B_2 for some prime $q \neq p$. Therefore, by Theorem 5.3, there is a $b(\underline{\underline{D}})$-group, A_1 say, with Sylow q-subgroups of greater class than those of B_2 . Now A_1 appears as a $((B_1,T)$-pass group)-pass group and, since B_1 has no quotient isomorphic with B_2 whereas T has, we have $T \neq K_1$ hence $|A_1| \leq |A| < |B_1|$. Therefore $A_1 \in b(\underline{\underline{D}}_1) = b^*(B_2)$. In particular A_1 is a B_2-pass group and so it cannot have Sylow q-subgroups of greater class than those of B_2 . This contradiction gives $b(\underline{\underline{D}}) = b^*(B_1)$ and concludes the proof. $\square$

Now we have the promised examples of $\underline{D}$-classes with constant depth n for any $n \in \mathbb{N}$.

(7.5) Example

Let H_i be a cyclic q-group of order q^i for $i = 0,1,.. n-1$. Let V_i be a faithful and irreducible $GF(p).H_i$-module and $G_i = [V_i]H_i$. Let $\underline{\underline{D}}_i = (h(G_i))^0$. Then $b(\underline{\underline{D}}_i)\backslash(G_i) = b(\underline{\underline{D}}_{i-1})$. Now $\underline{\underline{D}}_0 = h(Z_p) = \underline{\underline{Q}}^p$ is special and therefore each $\underline{\underline{D}}_i$ is special by Lemma 6.8. In particular, $\underline{\underline{D}}_{n-1}$ is special and $|b(\underline{\underline{D}}_{n-1})| = n$, therefore $\underline{\underline{D}}_{n-1}$ has constant depth n by Lemma 7.2. □

§8. $\underline{D}$-class with constant depth.

Here we present our main result concerning $\underline{D}$-classes, a characterization of those $\underline{D}$-classes with constant depth in terms of their avoidance classes.

(8.1) Lemma

If a $\underline{D}$-class $\underline{\underline{D}}$ has constant depth n for some $n \in \mathbb{N}$, then $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p.

Proof This follows immediately from Example 4.5. □

The following two results will be used in eliminating a possible counterexample to our main theorem.

(8.2) Proposition

Let $\underline{\underline{D}}$ be a $\underline{D}$-class with constant depth n and suppose that (*) holds for all $m < n$. Then $b(\underline{\underline{D}})$ only has generators of type I.

(*): A $\underline{D}$-class $\underline{\underline{X}}$ has constant depth m if and only if $\underline{\underline{X}}$ is special and $|b(\underline{\underline{X}})| = m$.

Proof Let $\underline{\underline{D}}$ be as described. We first handle the case where $b(\underline{\underline{D}})$ has more than one type I-generator.

Suppose $b(\underline{\underline{D}})$ has a type II-generator, J say. Let H be a $b(\underline{\underline{D}})$-group such that J appears as a quotient of some subgroup of a stabilizer of H (H exists by Lemma 6.16). Let G be a generator for $b(\underline{\underline{D}})$ having H as a G-pass group. Let A be a type I-generator distinct from G. Then $b(\underline{\underline{D}})\backslash(A)$ is a $\underline{D}$-boundary, $b(\underline{\underline{X}})$ say. Now $(H,J) \leq b(\underline{\underline{X}}) \subseteq b(\underline{\underline{D}})$ and so J is a type II-generator for $b(\underline{\underline{X}})$. Since $\underline{\underline{X}} >> \underline{\underline{D}}$, we have that $\underline{\underline{X}}$ has constant depth r for some $r < n$. Then by (*) it follows that $\underline{\underline{X}}$ is special. Therefore all generators of $b(\underline{\underline{X}})$ are of type I by Lemma 6.17. In particular this contradicts the fact that J is a type II generator for $b(\underline{\underline{X}})$.

We now assume that $b(\underline{\underline{D}})$ has just one type I-generator, A say. Again we suppose that $b(\underline{\underline{D}})$ has a type II-generator J and look for a contradiction. Let H and G be as before. Then $\ell(G) \geq \ell(H) > \ell(J)$ and $\gamma_q(G) \geq \gamma_q(H) \geq \gamma_q(J)$ for all primes q. Since A is the only type I-generator, we have $\ell(A) \geq \ell(B)$ and $\gamma_q(A) \geq \gamma_q(B)$ for all $b(\underline{\underline{D}})$-groups B and all primes q.

Suppose there exists a group R in $a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ such that the only $b(\underline{\underline{D}})$-group appearing as a quotient of R is A. In other words $R \in h(b(\underline{\underline{D}})\backslash(A))$. Let $D \in Proj_{\underline{\underline{D}}}(S)$ for $S \in Stab(R)$. Since $\underline{\underline{D}}$ has constant depth n, we have $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p by Lemma 8.1. We now apply Lemma 4.1 to see that some $b(\underline{\underline{D}})$-group has Sylow q-subgroups with class greater than $\gamma_q(A)$ for some prime q

distinct from p. Therefore some generator for $b(\underline{\underline{D}})$ must have Sylow q-subgroups of class greater than $\gamma_q(A)$. This contradiction allows us to assume that no such group R exists.

Let $b(\underline{\underline{X}}) = b(\underline{\underline{D}})\backslash(A)$. Then $\underline{\underline{X}}$ is a $\underline{D}$-class and we show that $\underline{\underline{D}}$ is $\underline{D}$-maximal in $\underline{\underline{X}}$. Suppose not, then there exists a $\underline{D}$-class $\underline{\underline{Y}}$ with $\underline{\underline{X}} \underset{\neq}{>} \underline{\underline{Y}} \underset{\neq}{>} \underline{\underline{D}}$. Then $b(\underline{\underline{X}}) \subset a(\underline{\underline{Y}}) \subset a(\underline{\underline{D}})$. If B is a $b(\underline{\underline{X}})$-group, then B has a quotient, B/N say, in $b(\underline{\underline{Y}})$. In turn B/N has a quotient in $b(\underline{\underline{D}})$. Since $B \in b(\underline{\underline{D}})$ and $b(\underline{\underline{D}})$ is a boundary, we have $N = 1$ and $B \in b(\underline{\underline{Y}})$. Therefore $b(\underline{\underline{X}}) \subset b(\underline{\underline{Y}}) \subset a(\underline{\underline{D}})$. If $T \in b(\underline{\underline{Y}})\backslash b(\underline{\underline{X}})$, then the only $b(\underline{\underline{D}})$-group appearing as a quotient of T is A, since $b(\underline{\underline{Y}})$ is a boundary. We have already seen that such a group cannot exist. Therefore no such $\underline{D}$-class $\underline{\underline{Y}}$ exists and so $\underline{\underline{D}}$ is $\underline{D}$-maximal in $\underline{\underline{X}}$. Then $\underline{\underline{X}}$ must have constant depth $n-1$. Now (*) yields $|b(\underline{\underline{X}})| = n-1$ and so $|b(\underline{\underline{D}})| = n$. Applying Proposition 7.3 gives that $\underline{\underline{D}}$ is special and the final contradiction comes from Lemma 6.17. □

(8.3) <u>Lemma</u>

Suppose n is minimal subject to there being a $\underline{D}$-class, $\underline{\underline{D}}$ say, with constant depth n and having a boundary consisting of less than n isomorphism classes. Then

(i) $b(\underline{\underline{D}})$ has only generators of type I.

(ii) For each generator H, there is a group $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$

having a quotient isomorphic with H and satisfying $G \in b^*(G)$. Furthermore, if G is chosen minimally with respect to these properties, then $g(\underline{\underline{D}})\backslash(H) \subseteq b^*(G)$.

<u>Proof</u> Let $\underline{\underline{D}}$ be as above with $|b(\underline{\underline{D}})| = r < n$.

(i) It follows by the minimality of n and Lemma 6.18 that if $m < n$ and $\underline{\underline{X}}$ is a $\underline{D}$-class with constant depth m , then $|b(\underline{\underline{X}})| = m$. Then Proposition 7.3 and Lemma 6.19 together show that $\underline{\underline{X}}$ is special. Now the conditions of Proposition 8.2 are satisfied and we conclude that all $b(\underline{\underline{D}})$ generators are of type I.

(ii) Let H be a generator for $b(\underline{\underline{D}})$. Then by (i) H is type I. Therefore $b(\underline{\underline{D}})\backslash(H)$ is a $\underline{D}$-boundary, $b(\underline{\underline{X}})$ say. Now $|b(\underline{\underline{X}})| = r-1$ and so from the choice of $\underline{\underline{D}}$ we conclude that $\underline{\underline{X}}$ has constant depth $r-1$. Consider a $\underline{D}$-maximal refinement of the chain $\underline{\underline{S}} >> \underline{\underline{X}} >> \underline{\underline{D}}$. There exists a $\underline{D}$-class $\underline{\underline{Y}}$ with $\underline{\underline{X}} \underset{\neq}{>>} \underline{\underline{Y}} \underset{\neq}{>>} \underline{\underline{D}}$ and hence $b(\underline{\underline{X}}) \subset a(\underline{\underline{Y}}) \subset a(\underline{\underline{D}})$. Let B be a $b(\underline{\underline{X}})$-group. Then B has a quotient in $b(\underline{\underline{Y}})$ which in turn has a quotient in $b(\underline{\underline{D}})$. Since $b(\underline{\underline{D}})$ is a boundary, we have $B \in b(\underline{\underline{Y}})$. Therefore $b(\underline{\underline{X}}) \subset b(\underline{\underline{Y}})$. Let $G \in b(\underline{\underline{Y}})\backslash b(\underline{\underline{X}})$. Since $b(\underline{\underline{Y}})$ is a boundary and $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$, we conclude that G has a quotient isomorphic with H . Since $G \in b(\underline{\underline{Y}})$, we have $G \in b^*(G)$ by Lemma 5.26.

We now choose G minimally such that G has a quotient isomorphic with H , $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ and $G \in b^*(G)$. Suppose there is a generator T distinct from H with $T \notin b^*(G)$. Consider an $h(b(\underline{\underline{D}})\backslash(T))$-projector L of $K \in \mathrm{Stab}(G)$. By Theorem 5.3, all (G,L)-pass groups lie in $a(\underline{\underline{D}})$. Now T does not appear as a G-pass group otherwise T would have a proper quotient in $b^*(G) \subset a(\underline{\underline{D}})$ and hence a proper quotient in $b(\underline{\underline{D}})$ contrary to the definition of a boundary. Suppose all (G,L)-pass groups lie in $b(\underline{\underline{D}})$. Then $G \in a(h(b(\underline{\underline{D}})\backslash(T)))\backslash(b(\underline{\underline{D}})\backslash(T))$ by Lemma 2.24 and so $h(b(\underline{\underline{D}})\backslash(T))$ is not special by Theorem 6.5. We already have by (i) that $h(b(\underline{\underline{D}})\backslash(T))$ is special. Therefore some (G,L)-pass group, R say, must lie in $a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. By the choice of G , we have $H \notin p(G)$ and hence $H \notin p(R)$. Let $S \in \mathrm{Stab}(R)$ and let J be an $h(b(\underline{\underline{D}})\backslash(H))$-projector of S . Let A be an (R,J)-pass group. Suppose that $A \notin b(\underline{\underline{D}})$. Then A has a quotient isomorphic with H and $A \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. By the minimal choice of G , we have $A \notin b^*(A)$. Therefore A has a proper quotient, C say, in $b^*(A)$. Now $b^*(A) \subset a(\underline{\underline{D}})$ so C has a quotient isomorphic to H since $C \in Q(J) \subseteq h(b(\underline{\underline{D}})\backslash(H))$. Then $C \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ and C has a proper quotient isomorphic with H . Also, since $C \in b^*(A)$ it follows from Lemma 5.26 that $C \in b^*(C)$. This contradicts the choice of G and hence we may assume that all (R,J)-pass groups lie in $b(\underline{\underline{D}})$. Since $H \notin p(R)$ it follows that $R \in a(h(b(\underline{\underline{D}})\backslash(H)))\backslash(b(\underline{\underline{D}})\backslash(H))$. Furthermore $T \notin p(G)$ implies that $T \notin p(R)$ and hence $T \notin b^*(R)$ by Lemma 2.24. Suppose $R \notin b^*(R)$. Then R has a proper quotient,

B say, in $b^*(R) \subseteq a(\underline{\underline{D}})$. Now Lemma 5.26 shows that $B \in b^*(B)$ and $B \in p(R)$. Since $h(b(\underline{\underline{D}})\backslash(H))$ is a $\underline{D}$-class, we have $B \in a(h(b(\underline{\underline{D}})\backslash(H)))\backslash(b(\underline{\underline{D}})\backslash(H))$ by Theorem 5.3 since $T \in (Q-1)(B)$. Now we have that $h(b(\underline{\underline{D}})\backslash(H))$ is not special by Theorem 6.5 contrary to what we have already established in (i). Therefore $R \in b^*(R)$. Now since $R \in a(h(b(\underline{\underline{D}})\backslash(H)))\backslash(b(\underline{\underline{D}})\backslash(H))$ we have $h(b(\underline{\underline{D}})\backslash(H))$ is not special by Theorem 6.5 and (i) is contradicted once again. This contradiction completes the proof. □

As promised earlier we now prove a stronger version of Lemma 6.18.

(8.4) Theorem

If $\underline{\underline{D}}$ is a $\underline{D}$-class with constant depth n in $\underline{D}$, then $|b(\underline{\underline{D}})| = n$.

Proof The result is true if $\underline{\underline{D}}$ has constant depth one by Lemma 6.18. We suppose the result is false and choose $\underline{\underline{D}}$ such that $\underline{\underline{D}}$ has constant depth n and $|b(\underline{\underline{D}})| \neq n$ and moreover n is minimal for such a class to exist. By Lemma 6.18 we have $|b(\underline{\underline{D}})| < n$ and $b(\underline{\underline{D}}) = b_p(\underline{\underline{D}})$ for some prime p by Lemma 8.1.

Let $\underline{\underline{T}} = (T \in g(\underline{\underline{D}}) : \ell(T) = \max\{\ell(G) : G \in b(\underline{\underline{D}})\})$ and $\underline{\underline{C}} = (G \in (a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})) : G$ has a quotient in $\underline{\underline{T}}$ and $G \in b^*(G))$. Then $\underline{\underline{C}} \neq \phi$ by Lemma 8.3. Let G be of minimal order in $\underline{\underline{C}}$ and let T be a $\underline{\underline{T}}$-quotient of G. Let $\ell = \ell(T)-1$. Then by choice of $\underline{\underline{D}}$ and Lemma 8.3 we have $g(\underline{\underline{D}})\backslash(T) \subseteq b^*(G)$. Let $K \in \mathrm{Stab}(G)$ and $X \in \mathrm{Proj}_{\underline{\underline{D}}}(K)$. Lemma 4.1 yields that $L_{\ell-1}(T)/L_{\ell}(T)$ is a q-group for some prime q distinct from p and also that there exists a $b(\underline{\underline{D}})$-group having Sylow q-subgroups with class greater than $\gamma_q(T)$. We establish some properties of G.

(1) $\underline{\ell(G) = \ell+3 \text{ and } Q(G) \cap b(\underline{\underline{D}}) = (T)}$

<u>Proof</u> Let $H \in \mathrm{Proj}_{\underline{\underline{N}}^{\ell+1}}(K)$ and $J = F(K).H$. Since $O_p(K) = 1$ and $F(K) \leq F(J)$, if $p \,|\, |F(J)|$, then $Z(O_p(J))$ centralizes $F(J)$ and hence $F(K)$. Therefore $Z(O_p(J)) \leq F(K)$ by Lemma 1.2. Thus $Z(O_p(J)) = 1$ and hence $O_p(J) = 1$. Since $T \in Q(G)$ and $\ell(T) = \ell+1$, we have $T \in Q(H)$ and hence $T \in Q(J)$. Let $L \in \mathrm{Proj}_{h(b(\underline{\underline{D}})\backslash(T))}(J)$. Then $F(K) \leq L$ and L has a quotient isomorphic with T but no other $b(\underline{\underline{D}})$-quotients. Then if $\underline{\underline{F}}$ is the formation given by $\underline{\underline{F}} = (S \in \underline{\underline{S}} : [H/K]\,\mathrm{Aut}_S(H/K) \not\gtrsim T$ for all p-chief factors H/K of $S)$, there is a (G,L)-pass group, R say, with $R/F(R) \notin \underline{\underline{F}}$ since $L \notin \underline{\underline{F}}$. Now $R/F(R) \in Q(L) \subseteq h(b(\underline{\underline{D}})\backslash(T))$. Theorem 5.3 yields that $R \in a(\underline{\underline{D}})$. Since $\ell(J) \leq \ell+2$ and $T \in Q(J/F(J))$ we have $\ell(J) = \ell+2$ and so $\ell(R) = \ell+3$. Furthermore $Q(R) \cap b(\underline{\underline{D}}) = (T)$.

We show $R \in b^*(R)$ and then the minimality of G yields $G = R$ hence proving (1). Because $G \in b^*(G)$ and $T \in Q(G)$, we have $T \notin b^*(G)$. If $T \in p(G)$, then T has a proper quotient isomorphic with a $b^*(G)$-group but $b^*(G) \subseteq a(\underline{\underline{D}})$ and so T has a proper quotient in $b(\underline{\underline{D}})$ giving a contradiction of $T \in b(\underline{\underline{D}})$ since $b(\underline{\underline{D}})$ is a Schunck boundary. Thus $T \notin p(G)$ and so $T \notin p(R)$ by Lemma 2.13. In particular, $T \notin b^*(R)$.

Suppose $R \notin b^*(R)$. Then R has a proper quotient in $b^*(R) \subseteq a(\underline{\underline{D}})$. In this case let Q be a $b^*(R)$-quotient of R. Then $F(Q)$ is a p-group and so $Q \in Q(R/F_2(R)) \subseteq h(b(\underline{\underline{D}})\backslash(T))$. Also $Q \in a(\underline{\underline{D}})$ since $b^*(R) \subseteq a(\underline{\underline{D}})$. Hence $T \in (Q-1)(Q)$ and so $\ell(Q) \geq \ell(T) + 2 = \ell+3$. This contradicts $\ell(R) = \ell+3$ and $Q \in Q\ (R/F_2(R))$ and so $R \in b^*(R)$. Thus (1) is proved.

Let X_q be a Sylow q-subgroup of X. Set $d = \max\{\gamma_q(B) : B \in b(\underline{\underline{D}})\}$ and write Γ for $\Gamma_d\ (X_q)$. Thus $\Gamma \leq Z(X_q)$. By Lemma 4.1 we have $\gamma_q(T) \leq d-1$. It follows from Lemma 8.3 that $\gamma_q(G) = d$ since $g(\underline{\underline{D}})\backslash(T) \subseteq b^*(G)$ and hence $\Gamma \neq 1$.

(2) $\underline{\Gamma \nleq F(K)}$

<u>Proof</u> We suppose $\Gamma \leq F(K)$ and obtain a contradiction. Suppose there exists a (G,X)-pass group R with $\gamma_q(R) = d$. Then $R \in b(\underline{\underline{D}})$ and $R/F(R)$ is isomorphic with X/C for some normal subgroup C of X. Therefore

$\Gamma C/C \le F(K)C/C \le F(X)C/C \le F(X/C) \cong F(R/F(R)) = F_2(R)/F(R)$

and hence $\gamma_q(R/F_2(R)) \le d-1$.

Let $B \in g(\underline{\underline{D}})$ satisfy $\gamma_q(B) = d$. Then $B \not\cong T$. We show that B is a (G,X)-pass group. Set $\underline{\underline{H}} = h(b(\underline{\underline{D}})\backslash(B))$. Since $B \in g(\underline{\underline{D}})\backslash(T) \subseteq b^*(G) \subseteq p(G)$ and $B/F(B) \in \underline{\underline{H}}$, we may view B as a (G,U)-pass group for some $\underline{\underline{H}}$-subgroup U of K . Let $Y \in \mathrm{Proj}_{\underline{\underline{D}}}(U)$. Then B is a (G,Y)-pass group. Now $\underline{\underline{D}} \in \underline{D}$ implies that $Y \le X^g$ for some $g \in K$ and so B is a $((G,X^g)$-pass group)-pass group. However B is a generator and all (G,X^g)-pass groups lie in $b(\underline{\underline{D}})$ hence B is a (G,X^g)-pass group and so also a (G,X)-pass group.

The arguments above now yield $\gamma_q(B/F_2(B)) \le d-1$. By Lemma 8.3 there is a group, E say, in $a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$ with $B \in Q(E)$ and $E \in b^*(E)$. Let $J \in \mathrm{Stab}(E)$. By Lemma 4.1 there exists a prime $r \ne p$ for which $\gamma_r(M) > \gamma_r(B)$ for some $b(\underline{\underline{D}})$-group M . Thus $r \ne q$. Let $D \in \mathrm{Proj}_{\underline{\underline{D}}}(J)$. We show $\gamma_q(D/F(D)) = d$.

We know that Γ centralizes X_q . If $\Gamma \le F(D)$, then Γ centralizes $F(J)$ since $F(J) \le F(D)$. Therefore $\Gamma \le F(J)$ by Lemma 1.2. Then $\gamma_q(J/F(J)) < d$ but B appears as a quotient of $J/F(J)$ and $\gamma_q(B) = d$. This contradiction yields $\Gamma \not\le F(D)$ and so $\gamma_q(D/F(D)) = \gamma_q(D) = d$. Now $D \in R_0$ (stabilizers of (E,D)-

pass groups) by Corollary 2.27. Therefore there is an (E,D)-pass group, A say, with $\gamma_q(A/F_2(A)) = d$. Let R be a generator for $b(\underline{\underline{D}})$ with $A \in p(R)$. Then $\gamma_q(R/F_2(R)) = d$. Now $R \not\gtrsim T$ since $\gamma_q(T) \leq d-1$. We have already established that R is a (G,X)-pass group and hence $\gamma_q(R/F_2(R)) \leq d-1$. This contradiction completes the proof of (2).

(3) $\underline{\Gamma \nleq F_2(K)}$

Proof We suppose that $\Gamma \leq F_2(K)$ and arrive at a contradiction. Set $\Delta = \Gamma^X.F(K)$. Now $F(K) \trianglelefteq X$ and so Δ is a normal subgroup of X . Since $F_2(K) \geq \Delta$ we have $\ell(\Delta) \leq 2$. Now $[\Delta/F(K), O_p(K/F(K))] \leq O_q(K/F(K)) \cap O_p(K/F(K)) = F(K)/F(K)$. It follows from (1) that $K/F(K) = O_p(K/F(K)).X/F(K)$. Since $\Delta/F(K) \trianglelefteq X/F(K)$ we conclude that $\Delta/F(K) \trianglelefteq K/F(K)$ and therefore $\Delta \trianglelefteq K$. Now $F(\Delta)$ char $\Delta \trianglelefteq K$ implies that $F(\Delta) \leq F(K) < \Delta$ and so $\ell(\Delta) = 2$.

Let V be a Δ-composition factor of $F(G)$ and set $C = C_\Delta(V)$, so $C \trianglelefteq \Delta$. Suppose $\Gamma^X \leq C$. Then, since $\Delta = \Gamma^X F(K)$, we have $\ell(\Delta/C) \leq \ell(F(K)) = 1$ and hence $L(\Delta) \leq C$. Now $L(\Delta)$ char $\Delta \leq K$ and so $L(\Delta) \trianglelefteq K$. Since $L(\Delta)$ is a normal, nilpotent subgroup of K, we have $L(\Delta) \leq F(K)$ and hence $O_p(L(\Delta)) = 1$. Applying Clifford's theorem (1.13), we see that $L(\Delta)$ acts non-trivially on every Δ-composition factor of $F(G)$. Therefore $L(\Delta)$ acts non-trivially on every X-composition factor of $F(G)$. It follows that Γ^X acts non-trivially on every X-composition factor of $F(G)$, and then so must Γ. Thus every (G,X)-pass group has Sylow q-subgroups of class d.

Since $\ell(X) = \ell(T)-1$ and X has a quotient isomorphic with $T/L_\ell(T)$, the fact that $L_{\ell-1}(T)/L_\ell(T)$ is a q-group implies that q divides $|L_{\ell-1}(X)|$ and hence $O_q(L_{\ell-1}(X)) \neq 1$. Therefore $O_q(L_{\ell-1}(X))$ must act non-trivially on some X-composition factor, V say, of $F(G)$. Set $S = [V]\, X/C_X(V)$. Then S is a $b(\underline{\underline{D}})$-group satisfying

(a) $\ell(S) = \ell(T)$

(b) q divides $\left|L_{\ell-1}(S)/L_\ell(S)\right|$

(c) $\gamma_q(S) = d$.

If $\bar{S}$ is a generator for $b(\underline{\underline{D}})$ having S as an $\bar{S}$-pass group, then $\bar{S}$ must also satisfy (a), (b) and (c).

Let $\underline{\underline{H}}$ be the Schunck class with boundary $b(\underline{\underline{D}})\backslash(\bar{S})$. Then $\underline{\underline{H}}$ is a $\underline{D}$-class by Lemma 8.3. We show that $\underline{\underline{D}}$ is maximal in $\underline{\underline{H}}$. Suppose $\underline{\underline{X}}$ is a $\underline{D}$-class satisfying $\underline{\underline{D}} <_{\neq}< \underline{\underline{X}} <_{\neq}< \underline{\underline{H}}$. Then $b(\underline{\underline{H}}) \subsetneq a(\underline{\underline{X}}) \subsetneq a(\underline{\underline{D}})$ and since $b(\underline{\underline{H}}) \subsetneq b(\underline{\underline{D}})$ we have $b(\underline{\underline{H}}) \subsetneq b(\underline{\underline{X}})$

appealing to the fact that $b(\underline{\underline{D}})$ is a boundary. Now any $(b(\underline{\underline{X}})\backslash b(\underline{\underline{H}}))$-group must have a quotient isomorphic with $\bar{S}$. Then by Lemma 4.1 there exists a $b(\underline{\underline{H}})$-group having Sylow q-subgroups of class at least $d+1$. This contradicts the choice of d and hence the above situation cannot occur. Thus $\underline{\underline{D}}$ is maximal in $\underline{\underline{H}}$.

Therefore $\underline{\underline{H}}$ has constant depth $n-1$, and so $|b(\underline{\underline{H}})| = n-1$ by the choice of n. This gives $|b(\underline{\underline{D}})| = n$ and we have a contradiction of the choice of $\underline{\underline{D}}$. This completes the proof of (3).

(4) <u>The Final Contradiction</u>

We summarise some of the properties of G.

(i) $G \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$

(ii) $\ell+1 \leq \ell(G) \leq \ell+3$

(iii) If $X \in \mathrm{Proj}_{\underline{\underline{D}}}(G)$, then $\ell(X) = \ell$ and q divides $|L_{\ell-1}(X)|$

(iv) $\gamma_q(G/F_3(G)) = d$

(v) $G/F_3(G) \in \underline{\underline{D}}$.

We now assume that G is a group with minimal order such that (i)-(v) are satisfied.

<u>Case I</u> : There is a normal subgroup N of K such that $K/N \in b(\underline{\underline{D}})$ and $\Gamma \leq N$.

Let B be a $b(\underline{\underline{D}})$-group isomorphic with K/N and let M/N be a p-chief factor of K corresponding to $F(B)$. Let Y/N complement M/N in K/N. Let $Y_q \in Syl_q(Y)$. Then, since $|K:Y|$ is a power of p, we have $Y_q \in Syl_q(K)$. By (v) we have $K = Y.F_2(K)$ since Y contains a $\underline{\underline{D}}$-projector, X, of K. Let $\Gamma = \Gamma_d\ (X_q)$ for some $X_q \in Syl_q(X)$. Therefore $\Gamma^Y \leq \Gamma^K \leq \Gamma^Y.F_2(K) \trianglelefteq K$, and so $\Gamma^K \leq \Gamma^Y.(F_2(K) \cap N) \leq Y$. From (iv) we have that $\ell(\Gamma^K) > 2$.

Now $\Gamma^K \leq \Gamma^Y(F_2(K) \cap N)$ implies that $\ell(\Gamma^Y(F_2(K) \cap N)) > 2$. Set $j = \ell(\Gamma^Y(F_2(K) \cap N))-1$ and $L_j = L_j(\Gamma^Y.(F_2(K) \cap N))$. Notice that since $\Gamma^Y.F_2(K) \trianglelefteq K$ and $N \trianglelefteq K$ we have $\Gamma^Y(F_2(K) \cap N) \trianglelefteq K$ and hence $L_j \trianglelefteq K$. Also $L_j \trianglelefteq Y$. Applying Clifford's Theorem (1.13) to L_j we see that L_j acts non-trivially on each L_j-composition factor of $F(G)$ since $L_j \trianglelefteq K$ and $L_j \in \underline{\underline{N}}$ imply that $L_j \leq F(K)$ and so L_j is not a p-group. Therefore L_j acts non-trivially on every Y-composition factor of $F(G)$. Let V be any Y-composition factor of $F(G)$. Set $C = C_Y(V)$. Then $C/C \neq L_j\ C/C \cong L_j(\frac{\Gamma^Y(F_2(K) \cap N)C}{C})$ and so $\ell(\Gamma^Y.(F_2(K) \cap N)C/C) > 2$. Now $X_qC/C \in Syl_q(XC/C)$ and $XC/C \in Proj_{\underline{\underline{D}}}(Y/C)$. Also $\Gamma_d(X_qC/C) = \Gamma C/C$.

We prove that $[V].(Y/C)$, which we denote by R, satisfies $\gamma_q(R/F(R) = d$. Suppose not. Then $\Gamma C/C \leq F_2(Y/C)$ and so $\Gamma^Y C/C \leq F_2(Y/C)$. However $(F_2(K) \cap N)C/C \leq F_2(Y)C/C \leq F_2(Y/C)$. Thus $\Gamma^Y(F_2(K) \cap N)C/C \leq F_2(Y/C)$ and hence $\ell(\Gamma^Y(F_2(K) \cap N)C/C) \leq 2$. This contradiction proves that R satisfies (iv). Now R is a (G,Y)-pass group and so $R \in a(\underline{\underline{D}})$ by Theorem 5.3.

By (iii) we may conclude that some (G,Y)-pass group R (now fixed) has a $\underline{\underline{D}}$-projector R_D with q dividing $|L_{\ell-1}(R_D)|$. (Notice that $R_D \cong XC/C$.) Thus R satisfies (iii) also. Since $R/F_3(R) \in Q(G/F_3(G))$, we have $R/F_3(R) \in \underline{\underline{D}}$ by (v). Hence R satisfies (v). Trivially R satisfies (ii). To show R satisfies (i)-(v) it only remains to show that $R \notin b(\underline{\underline{D}})$.

Suppose $R \in b(\underline{\underline{D}})$. Let P be a generator with $R \in p(P)$. Then $P \in \underline{\underline{T}}$ and $\gamma_q(P) = d$. We now appeal to Lemma 8.3(ii). Let W be an $(a(\underline{\underline{D}})\backslash b(\underline{\underline{D}}))$-group having a quotient isomorphic with P . Since q divides $|L_{\ell-1}(R)|$, we have that q also divides $|L_{\ell-1}(P)|$. Now Lemma 4.1 gives that $L_{\ell-1}(P)/L_{\ell}(P)$ is a q-group and that there is some $b(\underline{\underline{H}})$-group with Sylow q-subgroups of class greater than d . This contradicts the definition of d and hence we may conclude that $R \in a(\underline{\underline{D}})\backslash b(\underline{\underline{D}})$. Therefore R satisfies (i)-(v) and $|R| < |G|$ contrary to the choice of G . Thus Case I cannot occur.

Case II : If $N \trianglelefteq K$ and $K/N \in b(\underline{\underline{D}})$, then $\Gamma \nleq N$.

Recall that $F(K) \leq F(X)$ since $|K:X|$ is a power of p and $p \nmid |F(K)|$. Since $O_q(X)$ is a non-trivial q-group (because $q \mid |L_{\ell-1}(X)|$) , we have that $Z(O_q(X)) \neq 1$. Now $O_q(L_{\ell-1}(X))$ is a non-trivial normal subgroup of $O_q(X)$ and hence $Z(O_q(X)) \cap O_q(L_{\ell-1}(X)) \neq 1$. Furthermore $Z(O_q(X)) \cap O_q(L_{\ell-1}(X)) \leq C_X(F(X)) \leq C_K(F(K)) \leq F(K)$ by Lemma 1.2. Therefore $L_{\ell-1}(X) \cap O_q(K) \neq 1$. Therefore, writing L for $L_{\ell-1}(X) \cap O_q(K)$, we have $C_K(L) \geq C_K(O_q(K)) \geq \Gamma$ since $\Gamma \leq Z(X_q)$ for $X_q \in Syl_q(X)$. Since Case I does not occur, we have $K/C_K(O_q(K)) \in \underline{\underline{D}}$. Therefore $K = X.C_K(O_q(K)) = X.C_K(L)$.

Now $L \trianglelefteq X$ implies that $L \trianglelefteq K$. Now by Clifford's Theorem (1.13) we see that L acts non-trivially on every X-composition factor of $F(G)$. Thus all (G,X)-pass groups have nilpotent length $\ell+1$ and at least one, R say, must also have $\gamma_q(R) = d$. Since $R \in b(\underline{\underline{D}})$, there exists a generator $\bar{S}$ for $b(\underline{\underline{D}})$ with $R \in p(\bar{S})$. Then $\bar{S}$ clearly satisfies

(a) $\ell(\bar{S}) = \ell+1$

(b) q divides $|L_{\ell-1}(\bar{S})/L_\ell(\bar{S})|$

(c) $\gamma_q(\bar{S}) = d$.

The arguments now follow as at the end of (3) to give our final contradiction. □

This now gives us the following characterization of $\underline{D}$-classes with constant depth.

(8.5) Theorem

A $\underline{D}$-class $\underline{\underline{D}}$ has constant depth n if and only if $|b(\underline{\underline{D}})| = n$ and $\underline{\underline{D}}$ is special.

Proof By 7.2, 7.3 and 8.4. □

We now notice that no $\underline{D}$-class can be n-maximal in $\underline{H}$ for $n \geq 2$.

(8.6) Lemma

If $\underline{\underline{H}}$ is a Schunck class and $\underline{\underline{H}} \lneq\!< \underline{\underline{Q}}^p$, then $\underline{\underline{H}}$ is not 2-maximal in $\underline{H}$.

Proof Since $\underline{\underline{H}} << \underline{\underline{Q}}^p$, we have $(Z_p) = b(\underline{\underline{Q}}^p) \subset a(\underline{\underline{H}})$, and hence $Z_p \in b(\underline{\underline{H}})$. Let G be a $b(\underline{\underline{H}})$-group distinct from Z_p . Suppose G is non-cyclic. Then, by Lemma 1.20, $G \sim Z_p$ is a primitive group and it is easy to see that in fact $G \sim Z_p \in a(\underline{\underline{H}})$. Let $\underline{\underline{Y}}$ be the Schunck class with boundary $(G \sim Z_p, G)$. If G is cyclic, $G \cong Z_q$ say, then set $\underline{\underline{Y}} = h(E(p/q),G)$.

Now $b(\underline{\underline{Y}}) \subset a(\underline{\underline{H}})$ and hence $\underline{\underline{H}} << \underline{\underline{Y}} << h(G) << \underline{\underline{S}}$ is a proper chain of Schunck classes of length 3 and so $\underline{\underline{H}}$ is not 2-maximal in $\underline{H}$. □

(8.7) Corollary

If $\underline{\underline{D}}$ is a $\underline{D}$-class and $\underline{\underline{D}} \neq \underline{\underline{Q}}^p$ for all primes p , then $\underline{\underline{D}}$

is not n-maximal in $\underline{H}$ for any $n \in \mathbb{N}$.

Proof If $\underline{\underline{D}} \neq \underline{\underline{Q}}^p$ for all primes p , then $|b(\underline{\underline{D}})| \geq 2$ by Theorem 5.3. Refine the chain $\underline{\underline{S}} >> \underline{\underline{Q}}^p >> \underline{\underline{D}}$, where p is chosen such that $b_p(\underline{\underline{D}}) \neq \phi$, to a maximal chain of Schunck classes. If $\underline{\underline{D}}$ is n-maximal in $\underline{H}$ for some n , then this chain must contain a Schunck class $\underline{\underline{X}}$ such that $\underline{\underline{X}} << \underline{\underline{Q}}^p$ and $\underline{\underline{X}}$ is 2-maximal in $\underline{H}$. However, no such $\underline{\underline{X}}$ exists by Lemma 8.6 and so $\underline{\underline{D}}$ is not n-maximal in $\underline{H}$. □

§9. Minimal Schunck Classes

We begin with a necessary and sufficient condition for a Schunck class to be minimal and then give some examples of some minimal Schunck classes.

(9.1) Lemma

A Schunck class $\underline{\underline{H}}$ is minimal in $\underline{H}$ if and only if $h(b'(b(\underline{\underline{H}}) \cup (G))) = \underline{\underline{I}}$ for every primitive $\underline{\underline{H}}$-group G.

Proof Let $\underline{\underline{H}}$ be a minimal Schunck class and let G be any primitive $\underline{\underline{H}}$-group. Let $\underline{\underline{X}} = h(b'(b(\underline{\underline{H}}) \cup (G)))$. Then $b(\underline{\underline{H}}) \cup (G) \subseteq a(\underline{\underline{X}})$ by Proposition 3.5. Now $G \notin a(\underline{\underline{H}})$ hence $a(\underline{\underline{H}}) \neq a(\underline{\underline{X}})$. Therefore $\underline{\underline{H}} >_{\neq}> \underline{\underline{X}}$ by Theorem 2.19. The minimality of $\underline{\underline{H}}$ now yields $\underline{\underline{X}} = \underline{\underline{I}}$.

Conversely, suppose $\underline{\underline{H}}$ is a Schunck class which is not minimal. Then there exists a Schunck class, $\underline{\underline{Z}}$ say, with $\underline{\underline{H}} >> \underline{\underline{Z}} >> \underline{\underline{I}}$. Then $a(\underline{\underline{H}}) \subsetneq a(\underline{\underline{Z}})$. We show that $a(\underline{\underline{Z}})$ contains a primitive $\underline{\underline{H}}$-group.

Let G be minimal in $(a(\underline{\underline{Z}}) \setminus a(\underline{\underline{H}}))$. We first establish that $G \in b(\underline{\underline{Z}})$. If $G \in (a(\underline{\underline{Z}}) \setminus b(\underline{\underline{Z}}))$, then G has a proper quotient, B say, in $b(\underline{\underline{Z}})$. Now $|B| < |G|$ and so $B \in a(\underline{\underline{H}})$. Furthermore, if $L \in \mathrm{Proj}_{h(B)}(S)$ for $S \in \mathrm{Stab}(G)$, then all (G,L)-pass groups lie in $a(\underline{\underline{Z}})$ by Lemma 2.23 since $h(B) >> \underline{\underline{Z}}$. Hence all lie in $a(\underline{\underline{H}})$ by the minimality of G. It follows from Lemma 2.25 that $G \in a(\underline{\underline{H}})$. This contradiction gives $G \in b(\underline{\underline{Z}})$.

We now show that $G \in \underline{\underline{H}}$. If $G \notin \underline{\underline{H}}$, then G has a $b(\underline{\underline{H}})$-quotient, T say. Now $T \in b(\underline{\underline{H}}) \subset a(\underline{\underline{Z}})$ and so T has a quotient in $b(\underline{\underline{Z}})$. Therefore, since $G \notin a(\underline{\underline{H}})$ implies $T \not> G$, we have established that G has a proper $b(\underline{\underline{Z}})$-quotient. This contradicts $G \in b(\underline{\underline{Z}})$ and so we conclude that $G \in \underline{\underline{H}}$.

Now $a(\underline{\underline{H}}) \subsetneqq a(h(b'(a(\underline{\underline{H}}) \cup (G)))) \subseteq a(\underline{\underline{Z}})$ by Proposition 3.5 and Lemma 3.9(ii) since $h(G) \gg \underline{\underline{Z}}$ and $\underline{\underline{H}} \gg \underline{\underline{Z}}$. Therefore $h(b'(a(\underline{\underline{H}}) \cup (G))) \gg \underline{\underline{Z}}$ and hence, in particular, $h(b'(a(\underline{\underline{H}}) \cup (G))) \neq \underline{\underline{I}}$. □

(9.2) Example

Consider $\underline{\underline{N}}$, the class of nilpotent groups. Then $b(\underline{\underline{N}})$ consists of all primitive groups having nilpotent length 2. If $\underline{\underline{N}}$ is not minimal in $\underline{H}$, then, by Lemma 9.1, there is a primitive $\underline{\underline{N}}$-group, G say, such that $\underline{\underline{Z}} \neq \underline{\underline{I}}$ where $\underline{\underline{Z}} = h(b'(b(\underline{\underline{N}}) \cup (G))$. The only primitive nilpotent groups are the cyclic prime order groups. Let p be a prime and $G = Z_p$. Then $E(p/q) \in b(\underline{\underline{N}})$ for all primes q distinct from p . Therefore, since $b(\underline{\underline{N}}) \subseteq a(h(b'(b(\underline{\underline{N}}) \cup (G))))$ by Proposition 3.5, we have $Z_q \in a(\underline{\underline{Z}})$ for all primes q . Thus $b(\underline{\underline{Z}}) = (Z_r : r \in \mathbb{P})$ and $\underline{\underline{Z}} = \underline{\underline{I}}$. This shows that, in fact, $\underline{\underline{N}}$ is minimal in $\underline{H}$. □

In fact $\underline{\underline{N}}^{\ell}$ is minimal in $\underline{H}$ for all $\ell \in \mathbb{N}$. For a proof of this see Hawkes [5], 4.5.

(9.3) Proposition

If $\mathfrak{X}$ is a Schunck class strongly containing a Schunck class $\mathfrak{Y}$ which has a complement $\mathfrak{Y}'$ satisfying (*), then $\mathfrak{X}$ strongly contains an atom of $\underline{H}$.

(*) : There exists a $b(\mathfrak{Y}')$-group B such that
$h(b(\mathfrak{Y}') \setminus (B)) \wedge \mathfrak{Y} \neq \mathfrak{I}$.

Proof We suppose the result is false and take a Schunck class $\mathfrak{X}$ which does not contain an atom but which does satisfy our hypotheses. Then $\mathfrak{Y}$ does not strongly contain an atom. Set $b(\mathfrak{V}) = b(\mathfrak{Y}')\setminus(B)$. Then, since $b(\mathfrak{V}) \subset a(\mathfrak{Y}')$, it follows that $\mathfrak{V} >_{\neq}> \mathfrak{Y}'$ by Theorem 2.19. Set $\mathfrak{U} = \mathfrak{V} \wedge \mathfrak{Y}$. Then, by choice of $\mathfrak{Y}'$, we have $\mathfrak{U} \neq \mathfrak{I}$. Let $\mathfrak{L} = h(B)$. We show that $\mathfrak{L}$ complements $\mathfrak{U}$ in $\underline{H}$. We begin by showing that $\mathfrak{U} \vee \mathfrak{L} = \mathfrak{S}$. By Lemma 2.22, it is enough to show that $a(\mathfrak{U}) \cap a(\mathfrak{L}) = \phi$. Since $|b(\mathfrak{L})| = 1$, we have $a(\mathfrak{L}) = b(\mathfrak{L})$ by Lemma 2.34. Therefore it is enough to show that $B \notin a(\mathfrak{U})$. Suppose on the contrary that $B \in a(\mathfrak{U})$. Then since $\mathfrak{V} >> \mathfrak{U}$ implies that $b(\mathfrak{V}) \subseteq a(\mathfrak{U})$, we have $b(\mathfrak{Y}') = b(\mathfrak{V}) \cup (B) \subseteq a(\mathfrak{U})$. This yields $\mathfrak{Y}' >> \mathfrak{U}$ and then $\mathfrak{Y} >> \mathfrak{U}$ gives $\mathfrak{Y} \wedge \mathfrak{Y}' >> \mathfrak{U}$. However $\mathfrak{Y}'$ complements $\mathfrak{Y}$ and so $\mathfrak{Y} \wedge \mathfrak{Y}' = \mathfrak{I}$. This contradicts $\mathfrak{U} \neq \mathfrak{I}$ and so $B \notin a(\mathfrak{U})$ and hence $\mathfrak{U} \vee \mathfrak{L} = \mathfrak{S}$.

Now $\mathfrak{U} \wedge \mathfrak{L} = (\mathfrak{V} \wedge \mathfrak{Y}) \wedge \mathfrak{L} = \mathfrak{Y} \wedge (\mathfrak{V} \wedge \mathfrak{L})$. We prove that $\mathfrak{V} \wedge \mathfrak{L} = \mathfrak{Y}'$ and then conclude that $\mathfrak{U} \wedge \mathfrak{L} = \mathfrak{Y} \wedge \mathfrak{Y}' = \mathfrak{I}$. Together the statements

$\underline{\underline{L}} >> \underline{\underline{V}} \wedge \underline{\underline{L}}$, $\underline{\underline{V}} >> \underline{\underline{V}} \wedge \underline{\underline{L}}$, $\underline{\underline{L}} >> \underline{\underline{Y}}'$ and $\underline{\underline{V}} >> \underline{\underline{Y}}'$ imply $\underline{\underline{L}} >> \underline{\underline{V}} \wedge \underline{\underline{L}} >> \underline{\underline{Y}}'$ and $\underline{\underline{V}} >> \underline{\underline{V}} \wedge \underline{\underline{L}} >> \underline{\underline{Y}}'$. Therefore $b(\underline{\underline{Y}}) = (B) \cup b(\underline{\underline{V}}) \subseteq a(\underline{\underline{V}} \wedge \underline{\underline{L}}) \subseteq a(\underline{\underline{Y}}')$. Then by Theorem 2.19 again we have $\underline{\underline{Y}}' >> \underline{\underline{V}} \wedge \underline{\underline{L}} >> \underline{\underline{Y}}$ and so $\underline{\underline{Y}} = \underline{\underline{V}} \wedge \underline{\underline{L}}$.

Therefore $\underline{\underline{U}}$ complements $\underline{\underline{L}}$ in $\underline{H}$ and $\underline{\underline{I}} << \underline{\underline{U}} << \underline{\underline{Y}}$. Also $\underline{\underline{L}}$ complements $\underline{\underline{U}}$ and $|a(\underline{\underline{L}})| = |b(\underline{\underline{L}})| = 1$. Since $\underline{\underline{U}} << \underline{\underline{Y}}$ and $\underline{\underline{Y}}$ does not contain an atom of $\underline{H}$, it follows that $\underline{\underline{U}}$ cannot contain an atom. In particular $\underline{\underline{U}}$ is not an atom and so by Lemma 9.1 there is a primitive $\underline{\underline{U}}$-group G with $h(b'(b(\underline{\underline{U}}) \cup (G))) \neq \underline{\underline{I}}$. Let G_1 be such a group with smallest order. Set $\underline{\underline{U}}_1 = h(b'(b(\underline{\underline{U}}) \cup (G_1)))$ and define $\underline{\underline{U}}_2, \underline{\underline{U}}_3, \ldots\ldots$ by setting $\underline{\underline{U}}_{i+1} = h(b'(b(\underline{\underline{U}}_i) \cup (G_i)))$ for $i = 1,2,\ldots.$ where G_i is a smallest primitive $\underline{\underline{U}}_i$-group satisfying $h(b'(b(\underline{\underline{U}}_i) \cup (G_i))) \neq \underline{\underline{I}}$. Such a group G_i always exists since $\underline{\underline{U}}_i << \underline{\underline{U}}$ for each i and $\underline{\underline{U}}$ does not strongly contain any atoms.

Since $\underline{\underline{U}}_i << \underline{\underline{U}}$ and $\underline{\underline{U}} \wedge \underline{\underline{L}} = \underline{\underline{I}}$ we have $\underline{\underline{U}}_i \wedge \underline{\underline{L}} = \underline{\underline{I}}$ for each i . Now we show that $\underline{\underline{U}}_i \vee \underline{\underline{L}} = \underline{\underline{S}}$. Suppose $B \in a(\underline{\underline{U}}_i)$. Now $\underline{\underline{V}} >> \underline{\underline{U}} >> \underline{\underline{U}}_i$ and so $b(\underline{\underline{V}}) \subseteq a(\underline{\underline{U}}_i)$. Therefore $b(\underline{\underline{Y}}') = b(\underline{\underline{V}}) \cup (B) \subseteq a(\underline{\underline{U}}_i)$ and hence $\underline{\underline{Y}}' >> \underline{\underline{U}}_i$ but $\underline{\underline{Y}} >> \underline{\underline{U}} >> \underline{\underline{U}}_i$ and so $\underline{\underline{I}} = \underline{\underline{Y}} \wedge \underline{\underline{Y}}' >> \underline{\underline{U}}_i$. This contradiction yields $B \notin a(\underline{\underline{U}}_i)$ and hence $a(\underline{\underline{U}}_i) \cap a(\underline{\underline{L}}) = \phi$. Therefore $\underline{\underline{U}}_i \vee \underline{\underline{L}} = \underline{\underline{S}}$ and $\underline{\underline{U}}_i$ complements $\underline{\underline{L}}$ for each i .

Now $a(\underline{\underline{U}}_1) \subset a(\underline{\underline{U}}_2) \subset \ldots$ by Proposition 3.5. Set $b(\underline{H}) = (\bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i))^{\prec}$.

Suppose that $\bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i) \not\subseteq a(\underline{\underline{H}})$ and choose G with minimal order in $(\bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i) \backslash a(\underline{\underline{H}}))$. The definition of $\underline{\underline{H}}$ forces G to have a $b(\underline{\underline{H}})$-quotient, C say. Since $C \in b(\underline{\underline{H}})$, it follows that $C \in a(\underline{\underline{U}}_j)$ for some $j \in \mathbb{N}$. Let $k \geq j$ be chosen so that $G \in a(\underline{\underline{U}}_k)$. Thus $(G,C) \subseteq a(\underline{\underline{U}}_k)$. Let $S \in \mathrm{Stab}(G)$ and $J \in \mathrm{Proj}_{h(C)}(S)$. Since $h(C) >> \underline{\underline{U}}_k$ it follows that J contains a $\underline{\underline{U}}_k$-projector of G and hence all (G,J)-pass groups lie in $a(\underline{\underline{U}}_k)$ by Lemma 2.23. Because each (G,J)-pass group has order less than $|G|$, each lies in $a(\underline{\underline{H}})$. Then $G \in a(\underline{\underline{H}})$ by Lemma 2.25 since $h(C) >> \underline{\underline{H}}$. Hence we have established that $\bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i) \subseteq a(\underline{\underline{H}})$. Next we suppose that there is a group in $(a(\underline{\underline{H}}) \backslash \bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i))$ and choose G to be such a group with minimal order. Let T be a $b(\underline{\underline{H}})$-quotient of G. Trivially $T \not\cong G$ since $b(\underline{\underline{H}}) \subseteq \bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i)$. The minimality of G then yields $T \in \bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i)$ and hence there exists $j \in \mathbb{N}$ such that $T \in a(\underline{\underline{U}}_j)$. Let $S \in \mathrm{Stab}(G)$ and $L \in \mathrm{Proj}_{\underline{\underline{U}}_j}(S)$. We may apply Lemma 2.23 since $\underline{\underline{U}}_j >> \underline{\underline{H}}$ and conclude that all (G,L)-pass groups lie in $a(\underline{\underline{H}})$ and hence in $\bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i)$ by the minimality of G. There are only finitely many non-isomorphic (G,L)-pass groups, $(L_1,\ldots,L_n)$ say, since G

is finite. For $k = 1,\dots,n$ choose i_k such that $L_k \in a(\underline{\underline{U}}_{i_k})$. Let $i_0 = j$ and set $r = \max_{k=0,\dots,n} \{i_k\}$. Then $(T, L_1, \dots, L_n) \subseteq a(\underline{\underline{U}}_r)$. It follows from Lemma 2.25 that $G \in a(\underline{\underline{U}}_r)$ since $h(T) >> \underline{\underline{U}}_r$. Thus $G \in \bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i)$ and so $a(\underline{\underline{H}}) = \bigcup_{i=1}^{\infty} a(\underline{\underline{U}}_i)$.

If $B \in a(\underline{\underline{H}})$, then $B \in a(\underline{\underline{U}}_i)$ for some $i \in \mathbb{N}$ but we have already seen that this is not the case. Therefore $B \notin a(\underline{\underline{H}})$ and hence $\underline{\underline{H}} \neq \underline{\underline{I}}$.

We complete the proof by showing that $\underline{\underline{H}}$ is minimal in $\underline{H}$. If not, there exists a primitive $\underline{\underline{H}}$-group, H say, such that $\underline{\underline{W}} = h(b'(b(\underline{\underline{H}}) \cup (H))) \neq \underline{\underline{I}}$. Now $\underline{\underline{W}} = h(b'(a(\underline{\underline{H}}) \cup (H)))$ and $h(b'(b(\underline{\underline{U}}_i) \cup (H))) = h(b'(a(\underline{\underline{U}}_i) \cup (H)))$ by Lemma 3.8. Since $a(\underline{\underline{U}}_i) \cup (H) \subseteq a(\underline{\underline{H}}) \cup (H)$ and $\underline{\underline{W}} \neq \underline{\underline{I}}$, there is a prime p for which $Z_p \notin p(a(\underline{\underline{H}}) \cup (H))$ and so $Z_p \notin b(h(b'(b(\underline{\underline{U}}_i) \cup (H))))$. Therefore $h(b'(b(\underline{\underline{U}}_i) \cup (H))) \neq \underline{\underline{I}}$ for each i .

Since $H \in \underline{\underline{H}}$, we must have $H \in \underline{\underline{U}}_i$ for all $i \in \mathbb{N}$. But then H must be G_k for some $k \in \mathbb{N}$. Thus $H \in a(\underline{\underline{U}}_{k+1})$ by Proposition 3.5, hence $H \in a(\underline{\underline{H}})$. This contradicts $H \in \underline{\underline{H}}$. Therefore $\underline{\underline{H}}$ is minimal in $\underline{H}$ and $\underline{\underline{U}} >> \underline{\underline{H}}$. This contradiction completes the proof. □

(9.4) Corollary

If $\underline{\underline{X}}$ is a Schunck class strongly containing a Schunck class $\underline{\underline{Z}} \neq \underline{\underline{I}}$

which has a complement $\underline{\underline{Z}}'$ with finite boundary, then $\underline{\underline{X}}$ strongly contains an atom.

Proof Let $\underline{\underline{Z}}$ and $\underline{\underline{Z}}'$ be a described. Let $b(\underline{\underline{Z}}') = (B_1,\ldots,B_n)$ and set $\underline{\underline{B}}_0 = \underline{\underline{S}}$ and $\underline{\underline{B}}_i = h(B_1,\ldots,B_i)$ for $i = 1,\ldots,n$. Then $\underline{\underline{B}}_0 >> \underline{\underline{B}}_1 >> \ldots . >> \underline{\underline{B}}_n = \underline{\underline{Z}}'$. Now $\underline{\underline{Z}} \wedge \underline{\underline{B}}_0 = \underline{\underline{Z}} \neq \underline{\underline{I}}$ and $\underline{\underline{Z}} \wedge \underline{\underline{B}}_n = \underline{\underline{I}}$. Choose i such that $\underline{\underline{Z}} \wedge \underline{\underline{B}}_{i+1} = \underline{\underline{I}}$ but $\underline{\underline{Z}} \wedge \underline{\underline{B}}_i \neq \underline{\underline{I}}$. Now apply Proposition 9.3 with $\underline{\underline{Y}} = \underline{\underline{Z}}$, $\underline{\underline{Y}}' = \underline{\underline{B}}_{i+1}$ and $B = B_{i+1}$ to give the required result. □

(9.5) Corollary

If $\underline{\underline{X}}$ is a Schunck class strongly containing a Schunck class having finite non-empty characteristic, then $\underline{\underline{X}}$ strongly contains an atom.

Proof Let $\underline{\underline{Y}}$ be a Schunck class strongly contained in $\underline{\underline{X}}$ such that $|\mathrm{char}\,(\underline{\underline{Y}})|$ is finite. Therefore $b(\underline{\underline{Y}}) = b_p(\underline{\underline{Y}})$ for each prime p . Take a complement $\underline{\underline{Y}}'$ to $\underline{\underline{Y}}$ of the form described in (2.α) on page 41. Then $|b(\underline{\underline{Y}}')|$ is finite and Corollary 9.4 completes the proof. □

Recall that for an arbitrary class of groups $\underline{\underline{Y}}$, we denote by $\hat{\underline{\underline{Y}}}$ the smallest Schunck class containing (not necessarily strongly containing) $\underline{\underline{Y}}$.

(9.6) Proposition

If a Schunck class $\underline{\underline{X}}$ is an atom, then $\underline{\underline{X}} = \widehat{c(\underline{\underline{X}})}$.

Proof Set $\underline{\underline{Y}} = \widehat{c(\underline{\underline{X}})}$. We suppose that $\underline{\underline{X}}$ is an atom with $\underline{\underline{Y}} \subsetneq \underline{\underline{X}}$ and obtain a contradiction. Let G be a primitive group in $\underline{\underline{X}} \setminus \underline{\underline{Y}}$. Now G has no quotient in $b(\underline{\underline{X}})$ since it is an $\underline{\underline{X}}$-group. Suppose that T is a $b(\underline{\underline{X}})$-group having a quotient isomorphic with G . Then $G \in Q(c(\underline{\underline{X}})) \subseteq \underline{\underline{Y}}$. This contradiction yields that $b(\underline{\underline{X}}) \cup (G)$ is a Schunck boundary, $b(\underline{\underline{Z}})$ say. We have $\underline{\underline{Z}} \lneq \underline{\underline{X}}$ by Theorem 2.19 since $b(\underline{\underline{X}}) \subset b(\underline{\underline{Z}})$. Since $\underline{\underline{X}}$ is an atom, we conclude that $\underline{\underline{Z}} = \underline{\underline{I}}$ and hence $b(\underline{\underline{Z}}) = (Z_p : p \in \mathbb{P})$. Thus $b(\underline{\underline{X}}) = (Z_p : p \in \mathbb{P} \setminus \{q\})$ for some prime q . However then we see that $\underline{\underline{X}} \underset{\neq}{>} \underline{\underline{S}}_q$ and so $\underline{\underline{X}}$ is not an atom. This contradiction completes the proof. □

(9.7) Proposition

If $\underline{\underline{X}}$ is a Schunck class satisfying $\widehat{c(\underline{\underline{X}})} \subsetneq \underline{\underline{X}}$, then $\underline{\underline{X}}$ strongly contains an atom.

Proof Let $\underline{\underline{Y}} = \widehat{c(\underline{\underline{X}})}$. Let G be a primitive $(\underline{\underline{X}} \setminus \underline{\underline{Y}})$-group and let q be the prime divisor of $|F(G)|$. Let $K \in \mathrm{Stab}(G)$. For each prime p in $\mathrm{char}(\underline{\underline{X}}) \setminus \{q\}$ let W_p be an irreducible GF(p).G-module, faithful for G and such that $W_p|_K$ has a quotient isomorphic with the trivial K-module over GF(p) , such exists by Corollary 1.16. Let

$\underline{\underline{W}} = ([W_p]\, G : p \in \mathrm{char}(\underline{\underline{X}})\backslash\{q\})$. We claim $b(\underline{\underline{X}}) \cup \underline{\underline{W}}$ is a Schunck boundary. For if some $b(\underline{\underline{X}})$-group has a proper quotient in $\underline{\underline{W}}$, we obtain $G \in Q(c(\underline{\underline{X}})) \subseteq \underline{\underline{Y}}$. Also $[W_p]G$ has no proper quotient in $b(\underline{\underline{X}})$ since $G \in \underline{\underline{X}}$.

Now let $\underline{\underline{Z}} = h(\underline{\underline{W}} \cup b(\underline{\underline{X}})) \wedge h(G)$. Then $b(\underline{\underline{Z}}) = b'(b(\underline{\underline{X}}) \cup \underline{\underline{W}} \cup (G))$ by Lemma 2.22 and so $\mathrm{char}(\underline{\underline{Z}}) \subseteq \{q\}$ by the choice of the W_p's . If $\mathrm{char}(\underline{\underline{Z}}) = \{q\}$, then $\underline{\underline{Z}}$, and hence $\underline{\underline{X}}$, strongly contains an atom by Corollary 9.5. If, on the other hand, $\mathrm{char}(\underline{\underline{Z}}) = \phi$, then $\underline{\underline{Z}} = \underline{\underline{I}}$. Now $h(\underline{\underline{W}} \cup b(\underline{\underline{X}})) \vee h(G) = \underline{\underline{S}}$ since $a(h(G)) = (G)$ by Lemma 2.34, $G \notin a(h(\underline{\underline{W}} \cup b(\underline{\underline{X}})))$ as G has no quotient in $\underline{\underline{W}} \cup b(\underline{\underline{X}})$ and so $a(h(G)) \cap a(h(\underline{\underline{W}} \cup b(\underline{\underline{X}}))) = \phi = a(\underline{\underline{Z}})$ by Lemma 2.22. Therefore $h(\underline{\underline{W}} \cup b(\underline{\underline{X}}))$ has a complement, namely $h(G)$, with finite boundary and so by Corollary 9.4 $h(\underline{\underline{W}} \cup b(\underline{\underline{X}}))$, and hence $\underline{\underline{X}}$, strongly contains an atom. □

(9.8) <u>Corollary</u>

If $\underline{\underline{X}}$ does not strongly contain an atom, then $\underline{\underline{Y}} = \widehat{c(\underline{\underline{Y}})}$ for each Schunck class $\underline{\underline{Y}}$ strongly contained in $\underline{\underline{X}}$.

We now prove a result which has Wood's result of [12] as an immediate Corollary.

(9.8) <u>Theorem</u>

For each group G set $\rho_r(G)$ to be the set of primes dividing

$|L_{r-1}(G)/L_r(G)|$. If for a Schunck class $\underline{\underline{X}}$, there is some $r \in \mathbb{N}$ for which there is a bound on the set $\{|\rho_r(G) \cap \mathrm{char}(\underline{\underline{X}})| : G \in b(\underline{\underline{X}})\}$, then $\underline{\underline{X}}$ strongly contains an atom.

<u>Proof</u> Take r such that the set described is bounded and choose a $b(\underline{\underline{X}})$-group B with $|\rho_r(B) \cap \mathrm{char}(\underline{\underline{X}})|$ maximal among $b(\underline{\underline{X}})$-groups. Let $n = \ell(B) - 1$. Choose a prime p from $\mathrm{char}(\underline{\underline{X}})$ which does not divide $|B/L(B)|$. Recall that by Corollary 9.5 we may assume that $\mathrm{char}(\underline{\underline{X}})$ is infinite.

Let $G_0 = Z_p$ and $\underline{\underline{Y}}_0 = (G_0)$. For $j = 1,\ldots,n$ choose, if possible, a group G_{j-1} from $\underline{\underline{Y}}_{j-1} \setminus b(\underline{\underline{X}})$ and set $\underline{\underline{Y}}_j = ([V_q]G_{j-1} : V_q$ is an irreducible $GF(q).G_{j-1}$-module, faithful for G_{j-1} for each prime q satisfying $q \in \mathrm{char}(\underline{\underline{X}})$, $q \nmid |F(G_{j-1})|$ and $q \notin \rho_{j+1}(B))$.

If no such group G_{j-1} exists, then Case (i) below must apply for some i .

<u>Case (i)</u> $\underline{\underline{Y}}_i \subseteq b(\underline{\underline{X}})$ for some $i \in \{1,\ldots,n\}$.

Set $\underline{\underline{H}} = h(G_{i-1})$. Since $G_{i-1} \notin a(\underline{\underline{X}})$ and $a(\underline{\underline{H}}) = (G_{i-1})$ it follows from Lemma 2.22 that $a(\underline{\underline{X}} \vee \underline{\underline{H}}) = \phi$ and hence $\underline{\underline{X}} \vee \underline{\underline{H}} = \underline{\underline{S}}$. If $\underline{\underline{X}} \wedge \underline{\underline{H}} = \underline{\underline{I}}$, then our result follows from Corollary 9.4. If $\underline{\underline{X}} \wedge \underline{\underline{H}} \neq \underline{\underline{I}}$, then, by construction, it is clear that $\phi \neq \mathrm{char}(\underline{\underline{X}} \wedge \underline{\underline{H}}) \subseteq \rho_{i-2}(B) \cup \{t\}$

where t is the prime dividing $|F(G_{i-1})|$. In particular, $\mathrm{char}(\underline{\underline{X}} \wedge \underline{\underline{H}})$ is finite and so $\underline{\underline{X}} \wedge \underline{\underline{H}}$, and hence $\underline{\underline{X}}$, strongly contains an atom.

<u>Case (ii)</u> We assume that Case (i) does not apply. Then there is a $\underline{\underline{Y}}_n$-group H such that $H \in \underline{\underline{X}}$. Notice that $\ell(H) = n+1$. Let s be the prime divisor of $|F(H)|$. Now $\underline{\underline{Y}}_n \not\subseteq b_s(\underline{\underline{X}})$ since $\mathrm{char}(\underline{\underline{X}})$ is infinite and $\rho_{j+1}(B)$ is finite. Therefore we may assume the existence of $J \in (\underline{\underline{Y}}_n \backslash b_s(\underline{\underline{X}}))$. Let V be an irreducible $GF(w).(H\times J\times B)$-module faithful for $H\times J\times B$ where w is a prime not dividing $|F(H\times J\times B)|$, such exists by Lemma 1.17. Then $[V](H\times J\times B) \in \underline{\underline{X}}$ since $(H,B,J) \subseteq \underline{\underline{X}}$ and $\rho_r([V](H\times J\times B)) \not\supseteq \rho_r(B)$ does not allow $[V](H\times J\times B)$ to be a $b(\underline{\underline{X}})$-group. However $[V](H\times J\times B) \notin \widehat{c(\underline{\underline{X}})}$ and so $\underline{\underline{X}}$ strongly contains an atom by Proposition 9.6. □

(9.9) <u>Corollary</u> (Wood)

If $\underline{\underline{X}}$ is a Schunck class such that there is a bound on the nilpotent lengths of $b(\underline{\underline{X}})$-groups, then $\underline{\underline{X}}$ strongly contains an atom of $\underline{H}$.

<u>Proof</u> Choose r to be greater than the maximal nilpotent lengths of $b(\underline{\underline{X}})$-groups and apply Theorem 9.8. □

(9.10) Lemma

Let $\underline{\underline{X}}$ be a Schunck class. If there is a prime $q \in \text{char}\,\underline{\underline{X}}$ such that there is a bound on the set $\{\gamma_q(G) : G \in b(\underline{\underline{X}})\}$, then $\underline{\underline{X}}$ strongly contains an atom.

Proof Let $n = \max\{\gamma_q(G) : G \in b(\underline{\underline{X}})\}$. Choose a q-group Q with class $n+1$ such that there exists an irreducible $GF(p).Q$-module faithful for Q where $p \neq q$. Then $[V]Q \in \underline{\underline{X}} \setminus \widehat{c(\underline{\underline{X}})}$ and so $\underline{\underline{X}}$ strongly contains an atom by Proposition 9.6. $\square$

Bibliography

[1] DOERK, K., Über Homomorphe endlicher auflösbarer Gruppen. J. Algebra 30 (1974), 12-30.

[2] FÖRSTER, P., Charakterisierungen einiger Schunckklassen endlicher auflösbarer Gruppen I-IV

I. J. Algebra 55 (1978), 155-187.

II. Math. Z. 162 (1978), 219-234.

III. J. Algebra 62 (1980), 124-153.

IV. Arch. Math. 30 (1978), 247-252.

[3] GAGEN, T.M., Topics in Finite Groups. London Math. Soc. Lecture Notes Series 16. Cambridge University Press (1976).

[4] GASCHÜTZ, W., Lectures on Subgroups of Sylow Type in Finite Soluble Groups. Notes on Pure Mathematics 11. The Australian National University (1979).

[5] HAWKES, T.O., The family of Schunck classes as a lattice. J. Algebra 39 (1976), 527-550.

[6] HAWKES, T.O., An example in the theory of soluble groups. Proc. Cambridge Phil. Soc. 67 (1970), 13-16.

[7] HAWKES, T.O., Closure operations for Schunck classes. J. Austral. Math. Soc. 16, (1973), 316-318.

[8] HUPPERT, B., Endliche Gruppen I. Springer, Berlin Heidelberg (1967).

[9] KATTWINKEL, U., Die größte untergruppenabseschlossene Teilklasse einer Schunckklasse endlicher auflösbarer Gruppen. Arch. Math. 29 (1977), 337-343.

[10] SCHUNCK, H., H-Untergruppen in endlicher auflösbarer Gruppen Math. Z. 97 (1967), 326-330.

[11] WOOD, G.J., A lattice of homomorphs. Math Z 130 (1973), 31-37.

[12] WOOD, G.J., A note on strong containment in the theory of Schunck classes of finite soluble groups. J. London Math. Soc. (2) 13 (1976), 235-238.